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A SECOND COURSE IN ALGEBRA

BY

WEBSTER WELLS, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY

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PREFACE

In the preparation of this text the author acknowledges joint authorship with Robert L. Short, Technical High School, Cleveland.

A knowledge of the more elementary parts of algebra is presupposed. For this reason some definitions and rules for operation are assumed as already known to the pupil.

Attention is called to the generalization and bringing together of related topics. Chapter III is an example of this feature. Here all forms of the exponent are treated. This gives opportunity to regard the logarithm as a decimal exponent and to make the logarithmic operation laws intelligible. The introduction of all linear equations and inequalities in Chapter II shows their solution directly dependent upon the four fundamental operations. It is thought that the introduction of the idea of functionality and of algebraic forms taken directly from the calculus will be found helpful to those who expect to pursue the study of mathematics further.

The treatment of factoring is thorough and so taken up that Synthetic Division becomes the natural method for factoring many higher forms and for solving equations of higher degree.

It is hoped that the treatment of variation as a proportion will remove the reluctance with which most pupils approach that subject in connection with their work in science.

In scope this text is sufficient preparation for most courses in mathematics which require thorough knowledge of the operations of algebra.

WEBSTER WELLS.

CONTENTS

PART I

ALGEBRA

I. THE FUNDAMENTAL LAWS FOR ADDITION AND MULTIPLICATION

1. The Commutative Law for Addition.

If a man gains \$8, then loses \$3, then gains \$6, and finally loses \$2, the effect on his property will be the same in whatever *order* the transactions occur.

Then, the result of adding +\$8, − \$3, + \$6, and − \$2, will be the same in whatever order the transactions occur.

Then, omitting reference to the unit, the result of adding $+8$, -3, $+6$, and -2 will be the same in whatever order the numbers are taken.

This is the Commutative Law for Addition, which is:

The sum of any set of numbers will be the same in whatever order they may be added.

2. The Associative Law for Addition.

The result of adding $b+c$ to a is expressed $a+(b+c)$, which equals $(b+c)+a$ by the Commutative Law for Addition (§ 1).

But $(b+c)+a$ equals $b+c+a$; and $b+c+a$ equals $a+b+c$, by the Commutative Law for Addition.

Whence, $$a+(b+c)=a+b+c.$$

Then, **to add the sum of a set of numbers, we add the numbers separately.**

This is the Associative Law for Addition.

3. The Commutative Law for Multiplication.

The product of a set of numbers will be the same in whatever order they may be multiplied.

The *sign* of the product of any number of terms is independent of their order; hence, it is sufficient to prove the commutative law for *arithmetical numbers*.

Let there be, in the figure, a stars in each row, and b rows. We may find the entire number of stars by multiplying the number in each row, a, by the number of rows, b. Thus, the entire number of stars is $a \times b$.	a in a row. * * * ... * * * ... * * * ... . b rows.

We may also find the entire number of stars by multiplying the number in each vertical column, b, by the number of columns, a.

Thus, the entire number of stars is $b \times a$.

Therefore, $$a \times b = b \times a,$$

which is the law for the product of two positive integers.

Again, let c, d, e, and f be any positive integers.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{c \times e}{d \times f}$; for, to multiply two fractions, we multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

Then, $\frac{c}{d} \times \frac{e}{f} = \frac{e \times c}{f \times d}$; since the commutative law for multiplication holds for the product of two positive integers.

Hence, $\frac{c}{d} \times \frac{e}{f} = \frac{e}{f} \times \frac{c}{d}$; which proves the commutative law for the product of two positive fractions.

4. The Associative Law for Multiplication.

To multiply by the product of a set of numbers, we multiply by the numbers of the set separately.

The result of multiplying a by bc is expressed $a \times (bc)$, which equals $(bc) \times a$, by the Commutative Law for Multiplication.

$(bc) \times a$ equals bca, which equals abc by the Commutative Law for Multiplication.

Whence, $$a \times (bc) = abc.$$

This proves the law for the product of three numbers.

The Commutative and Associative Laws for Multiplication may be proved for the product of any number of arithmetical numbers.
(See the author's Advanced Course in Algebra, §§ 18 and 19.)

5. The Distributive Law for Multiplication.

The law is expressed $(a+b)c = ac + bc$.

We will now prove this result for all values of a, b, and c.

I. Let a and b have any values, and let c be a positive integer.

Then, $$(a+b)c = (a+b)+(a+b)+\cdots \text{ to } c \text{ terms}$$
$$= (a+a+\cdots \text{ to } c \text{ terms}) + (b+b+\cdots \text{ to } c \text{ terms})$$

(by the Commutative and Associative Laws for Addition),
$$= ac + bc.$$

II. Let a and b have any values, and let $c = \frac{e}{f}$, where e and f are positive integers.

Since the product of the quotient and divisor equals the dividend,
$$\frac{e}{f} \times f = e.$$

Then, $$(a+b) \times \frac{e}{f} \times f = (a+b) \times e = ae + be, \text{ by I.}$$

Whence, $$(a+b) \times \frac{e}{f} \times f = a \times \frac{e}{f} \times f + b \times \frac{e}{f} \times f.$$

Dividing each term by f, we have
$$(a+b) \times \frac{e}{f} = a \times \frac{e}{f} + b \times \frac{e}{f}.$$

Thus, the result is proved when c is a positive integer or a positive fraction.

III. Let a and b have any values, and let $c = -g$, where g is a positive integer or fraction.

$$(a+b)(-g) = -(a+b)g = -(ag+bg), \text{ by I and II,}$$
$$= -ag - bg = a(-g) + b(-g).$$

Thus, the distributive law is proved for all positive or negative, integral or fractional, values of a, b, and c.

II. ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, APPLICATIONS

6. Similar terms are those which do not differ at all or differ only in their coefficients.

7. Any factor of a product may be considered the **coefficient** of the product of the remaining factors.

8. To add two similar terms, write their coefficients with the proper sign and affix the common literal part.

Ex. 1. Find the sum of ax and bx.

$$ax + bx = (a + b)x.$$

Ex. 2. Find the sum of $3\,abcx$ and $-5\,mcx$.

$$3\,abcx + (-5\,mcx) = (3\,ab - 5\,m)cx.$$

This is equivalent to taking the common factor cx from the expression

$$3\,abcx - 5\,mcx.$$

9. To subtract two similar terms find what number added to the subtrahend will produce the minuend. The number added is called the **difference**. This is equivalent to changing the sign of the subtrahend and adding the result to the minuend.

Ex. 3. Subtract $3\,ax$ from $5\,ax$. $2\,ax$ added to $3\,ax$ is $5\,ax$. Hence $2\,ax$ is the difference.

Ex. 4. From $15\,m$ take $-8\,m$. Changing the sign (mentally) of $-8\,m$, we have $15\,m + 8\,m = 23\,m$.

The written work should appear in this form:

$$\begin{array}{r} 15\,m \\ -8\,m \\ \hline 23\,m \end{array}$$

10. Three laws enter into multiplication of monomials:

The law of signs.
The law of coefficients.
The law of exponents.

The product of two terms of like sign is positive; the product of two terms of unlike sign is negative.

To the product of the numerical coefficients annex the letters; giving to each an exponent equal to the sum of its exponents in the factors.

The same three laws enter into division, except that *quotient* takes the place of *product* and the exponent of the divisor is subtracted from the exponent of the same letter in the dividend. (Make a rule for division of monomials.) The reason for such rule follows readily when division is defined as the process of finding one of two numbers when their product and one of the numbers are given.

11. An **equation** is a statement that two numbers are equal.

12. If an equation is true for all finite values of the unknown numbers involved, it is an **identical equation** or **identity.**

13. If an equation is true only for a definite set of values of the unknown numbers involved, it is an **equation of condition.**

14. An equation may not be true for any values of the unknowns involved. It is then said to have no roots.

15. If when a number is substituted for an unknown in an equation, the equation becomes identical (§ 12) for that number, the equation is said to be satisfied.

The **roots** of an equation are the numbers which satisfy it. A *root* of an equation is also called a **solution of the equation.**

16. Some principles used in the solution of equations are a set of generally accepted truths called **axioms.** The axioms most frequently in use are:

1. If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.

2. If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.

3. If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.

4. If equal numbers be divided by the same number, or equal numbers except 0, the resulting numbers will be equal.

17. To solve an equation is to find its roots.

The following steps indicate the process:

$$\tfrac{2}{3}x - 5 = 15. \qquad (1)$$

Add 5 to each member, (Ax. 1)

$$\tfrac{2}{3}x = 15 + 5 = 20. \qquad (2)$$

Multiply each member by 3, (Ax. 3)

$$2x = 60. \qquad (3)$$

Divide each member by 2, (Ax. 4)

$$x = 30. \qquad (4)$$

18. Two equations are **equivalent** when every solution of the one is a solution of the other.

Thus equations (1), (2), (3), (4) are equivalent.

The axioms of algebra enable us to transform an equation into an equivalent one which may be more easily solved than the given one.

EXERCISE 1

1. Add $3a - 2b + 5c$, $b - 9a - 11c$, $3c + b - 2a$, $b - c - a$.

2. From the sum of $7x - 8y + 4z$ and $-2x + 5z + y$ take the sum of $x - y - z$ and $y + z - 9x$.

3. Add $3(m+n) - 5s + t$; $-8(m+n) + 4t - 11s$; $8s - 9(m+n) - 5t$; $6(m+n) - 4s + 3t$.

4. From $\frac{2}{3}p - \frac{5}{2}q + r$ take the sum of $\frac{1}{2}p + \frac{1}{3}q + \frac{2}{9}r$ and $\frac{1}{7}p - \frac{3}{4}q - \frac{1}{2}r$.

5. Subtract $ax + by + cz$ from $m^2x - y + dz$.

6. Subtract $(c-d)x - (c+d)y$ from $(2c+5d)x + (4c-3d)y$.

7. Take $mv + x$ from $md - x^2$.

8. From $4ab^2c + 5ab(c+d) - 9a^2bc^3$ take $(3a+5)b^2c - ab(c+d)$.

9. Simplify $(x^3 - 4x^2 + 5x - 1) - (2x^3 + 5x^2 - x - 7) + (x^3 + 2x^2 - 3x + 2)$.

10. Simplify $(x+1)(x-2)(x-3) - (x-2)^2 + (x^3 - 1)$.

11. Simplify $(x+y)^4 - (x-y)^4$.

12. Simplify $[4x^2 - (2x+5)][2x^2 - (x-3)]$.

13. Multiply $4x^2 + xy - y^2$ by $3x^2 - 5xy + 4y^2$.

14. Multiply $ax + by + cz$ by $bx - ay + cz$.

15. Multiply $4(m+n)^2 - 5(m+n) + 7$ by $(m+n)^2 + 2(m+n) + 1$.

16. Multiply $x^{2a+1} + x^a y^b + y^{2b}$ by $x^a - y^b$.

17. Expand $(4a + 3b)^2(4a - 3b)^2$.

18. Multiply $\frac{1}{3}a^2 - \frac{1}{4}ab + \frac{3}{5}b^2$ by $-\frac{2}{7}a + \frac{1}{3}b$.

19. Multiply $a^{2g} + a^g b^e + b^{2e}$ by $a^{2g} - a^g b^e + b^{2e}$.

20. Multiply $x^2 - xy + y^2 - xz - yz + z^2$ by $x + y + z$.

21. Multiply $x^2 + ax + bx + ab$ by $x + c$.

22. Divide $6x^6 - 19x^5 + 12x^4 + 5x^3 + 4x^2 - 6x - 2$ by $2x^2 - 3x - 1$.

23. Divide $a^{12} + b^{12}$ by $a^4 + b^4$.

24. Divide $32m^5 - 243n^5$ by $2m - 3n$.

25. Divide $\frac{1}{125}a^3 + \frac{8}{27}b^3$ by $\frac{1}{5}a + \frac{2}{3}b$.

26. Divide $a^{6n} - b^{6n}$ by $a^{2n} + a^n b^n + b^{2n}$.

27. Divide $\frac{1}{6}x^3 + \frac{7}{36}x^2y + \frac{1}{8}y^3$ by $\frac{1}{3}x + \frac{1}{2}y$.

28. Divide $9r^2s^2 + 15r^4 - 38r^3s - 8s^4 - 26rs^3$ by $5r^2 + 4s^2 - rs$.

29. Divide $7m^{2x+4} - 8m^{x+2}n^{2x-1} - 12n^{4x-2}$ by $m^{x+2} - 2n^{2x-1}$.

30. Divide $x^3 + (a+b)x^2 - (6a^2 - 5ab)x + 6a^2b$ by $x + 3a$.

Solve the following equations and verify results:

31. $(x+2)(x-5) = x^2 - 4x - 4$.

32. $6(x-3)+5(4x-7)+1=0.$

33. $\frac{3}{2}v-4+\frac{5}{3}v-\frac{1}{5}v=\frac{7}{3}v-\frac{1}{5}.$

34. $\frac{2}{5}(3x-2)-\frac{1}{2}(3x-2)=\frac{1}{7}(3x-2)-17.$

35. $(y-4)(y+3)(y-2)=(y-1)^3-1.$

36. $\frac{6t+5}{15}-\frac{13}{21}=\frac{2t}{5}+\frac{t}{3}.$

37. $ab+ax+3b^2-2a^2=4bc-bx+cx-c^2-ac.$ Solve for x.

38. $y-e=-\frac{1}{m}(x-d).$ Solve for x.

39. $(a+b+c)(x-2a)-(x-c)(a+b)$
$=(a-b-c)^2-(a^2+b^2).$

40. $\frac{ax-b}{a}+\frac{bx-c}{b}+\frac{cx-a}{c}=0.$

19. It is sometimes convenient to indicate operations of addition and subtraction. For this purpose parentheses are used. The various forms of parentheses are: *parentheses* ( ), *braces* { }, *brackets* [ ], and the *vinculum* $\overline{\quad}$.

A *positive sign* before parentheses indicates that the number within is to be added. Hence, parentheses preceded by a + sign may be removed without changing the signs of the terms within.

Ex. $2a+3b+(3a-5b+c)=2a+3b+3a-5b+c.$

A *negative sign* before parentheses indicates that the number within the parentheses is to be subtracted. Hence, parentheses preceded by a − sign may be removed if the + signs of the terms within be changed to − and the − signs to + (§ 9).

Ex. 1. $5a+3b-(4a+7b)=5a+3b-4a-7b=a-4b.$

Ex. 2. $5a+3b-(-4a+7b)=5a+3b+4a-7b=9a-4b.$

If the expression contains two or more parentheses, one within the other, remove one at a time beginning with the inner parentheses.

Ex.

$$5a + \{3a - (5b + 2a)\} =$$
$$5a + \{3a - 5b - 2a\} =$$
$$5a + 3a - 5b - 2a =$$
$$6a - 5b.$$

EXERCISE 2

Simplify the following by removing the signs of aggregation, and then uniting similar terms :

1. $9m + (-4m + 6n) - (3m - n).$
2. $2x - 3y - [5x + y] + \{-8x - 7y\}.$
3. $4y^2 - 2x^2 - [-4x^2 - 7xy + 5y^2] + (8x^2 - 9xy).$
4. $3a^2 - 5ab - \{-4a^2 + 2ab - 9b^2\} - \overline{7a^2 - 6ab + b^2}.$
5. $5a - (7a - [9a + 4]).$
6. $7x - \{-8y - \overline{10x - 11y}\}.$
7. $6mn + 5 - ([-7mn - 3] - \{-5mn - 11\}).$
8. $2a - (-3b + c - \{a - b\}) - (3a + 2c - [-2b + 3c]).$
9. $37 - [41 - \{13 - (56 - \overline{28 + 7})\}].$
10. $9m - (3n + \{4m - [n - 6m]\} - [m + 7n]).$

11. In each of the above expressions find the value if $a = 1$, $b = -2$, $c = -3$, $m = 5$, $n = 2$, $x = -4$, $y = -1$.

20. A number may be enclosed in parentheses preceded by a + sign without changing the sign of its terms, but if a number is enclosed in parentheses preceded by a − sign, each plus term placed in parentheses is changed to *minus* and each *minus* term to *plus*.

EXERCISE 3

In each of the following expressions, enclose the last three terms in parentheses preceded by a − sign :

1. $a - b - c + d.$
2. $m^3 + 2m^2 + 3m + 4.$
3. $x + x^2y - xy^2 - y^3.$
4. $a^2 - 4b^2 + 12b - 9.$

5. $4x^2 - y^2 - 2yz - z^2$.

6. $a^2 + b^2 - c^2 + d^2$.

7. $x^2 - 2xy + y^2 + 3x - 4y$.

8. $n^4 - 5n^3 - 8n^2 + 6n + 7$.

DEGREE OF A RATIONAL EXPRESSION

21. A monomial is said to be **rational and integral** when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is rational and integral.

A polynomial is said to be rational and integral when each term is rational and integral; as $2x^2 - \frac{3}{4}ab + c^3$.

22. If a term has a literal portion which consists of a single letter with unity for its exponent, the term is said to be of the *first degree.*

Thus, $2a$ is of the first degree.

The **degree** of any rational and integral monomial (§ 21) is the number of terms of the first degree which are multiplied together to form its literal portion.

Thus, $5ab$ is of the *second* degree; $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is of the *fifth* degree; etc.

The degree of a rational and integral monomial equals the sum of the exponents of the letters involved in it.

Thus, ab^4c^3 is of the *eighth* degree.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

23. If a rational and integral monomial (§ 21) involves a certain letter, its *degree with respect to it* is denoted by its exponent.

If it involves two letters, its *degree with respect to them* is denoted by the sum of their exponents; etc.

Thus, $2\,ab^4x^2y^3$ is of the *second* degree with respect to x and of the *fifth* with respect to x and y.

24. An **Integral Equation** is one each of whose members is a rational and integral expression (§ 21); as,

$$4\,x - 5 = \tfrac{2}{3}\,y + 1.$$

A **Numerical Equation** is one in which all the known numbers are represented by Arabic numerals; as,

$$2\,x - 7 = x + 6.$$

25. If an integral equation (§ 24) contains one or more unknown numbers, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax - by = c$ is an equation of the *first* degree;

$x^2 + 4\,x = -2$, an equation of the *second* degree;

$2\,x^2 - 3\,xy^2 = 5$, an equation of the *third* degree; etc.

A **Linear**, or **Simple, Equation** is an equation of the first degree.

26. The equations of Exercise 1 were integral, first degree in one unknown number, linear.

THEOREMS IN REGARD TO EQUIVALENT EQUATIONS

27. If the same expression be added to both members of an equation, the resulting equation will be equivalent to the first.

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $$A + C = B + C, \qquad (2)$$

where C is any expression, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B (§ 15).

It then makes $A + C$ identically equal to $B + C$ (§ 16, 1).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $A + C$ identically equal to $B + C$.

It then makes A identically equal to B (§ 16, 2).

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The principle of § 16, 1, is a special case of the above.

28. The demonstration of § 27 also proves that

If the same expression be subtracted from both members of an equation, the resulting equation will be equivalent to the first.

The principle of § 16, 2, is a special case of this.

29. If the members of an equation be multiplied by the same expression, which is not zero, and does not involve the unknown numbers, the resulting equation will be equivalent to the first.

Let $$A = B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $$A \times C = B \times C, \qquad (2)$$

where C is not zero, and does not involve the unknown numbers, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B.

It then makes $A \times C$ identically equal to $B \times C$ (§ 16, 3).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $A \times C$ identically equal to $B \times C$.

It then makes A identically equal to B (§ 16, 4).

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The reason why the above does not hold for the multiplier zero is, that the principle of § 16, 4, does not hold when the divisor is zero.

The principle of § 16, 3, is a special case of the above.

30. If the members of an equation be multiplied by an expression which involves the unknown numbers, the resulting equation is, in general, not equivalent to the first.

Consider, for example, the equation $x + 2 = 3x - 4$. (1)

Now the equation

$$(x + 2)(x - 1) = (3x - 4)(x - 1), \qquad (2)$$

which is obtained from (1) by multiplying both members by $x-1$, is satisfied by the value $x=1$, which does not satisfy (1).

Then (1) and (2) are not equivalent.

Thus it is never allowable to multiply both members of an integral equation by an expression which involves the unknown numbers; for in this way additional solutions are introduced.

31. If the members of an equation be divided by the same expression, which is not zero, and does not involve the unknown numbers, the resulting equation will be equivalent to the first.

Let $$A=B \qquad (1)$$

be an equation involving one or more unknown numbers.

To prove the equation $$\frac{A}{C}=\frac{B}{C}, \qquad (2)$$

where C is not zero, and does not involve the unknown numbers, equivalent to (1).

Any solution of (1), when substituted for the unknown numbers, makes A identically equal to B.

It then makes $\frac{A}{C}$ identically equal to $\frac{B}{C}$ (§ 16, 4).

Then it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes $\frac{A}{C}$ identically equal to $\frac{B}{C}$.

It then makes A identically equal to B.

Then it is a solution of (1).

Therefore, (1) and (2) are equivalent.

The principle of § 16, 4, is a special case of the above.

32. If the members of an equation be divided by an expression which involves the unknown numbers, the resulting equation is, in general, not equivalent to the first.

Consider, for example, the equation

$$(x+2)(x-1)=(3x-4)(x-1). \qquad (1)$$

Also the equation $$x+2=3x-4, \qquad (2)$$

which is obtained from (1) by dividing both members by $x-1$.

Now equation (1) is satisfied by the value $x=1$, which does not satisfy (2).

Then (1) and (2) are not equivalent.

It follows from this that it is never allowable to divide both members of an integral equation by an expression which involves the unknown numbers; for in this way solutions are lost.

33. If both members of a fractional equation be multiplied by the L. C. M. of the given denominators, the resulting equation is in general equivalent to the first.

Let all the terms be transposed to the first member, and let them be added, using for a common denominator the L. C. M. of the given denominators.

The equation will then be in the form

$$\frac{A}{B} = 0. \qquad (1)$$

We will now prove the equation

$$A = 0, \qquad (2)$$

which is obtained by multiplying (1) by the L. C. M. of the given denominators, equivalent to (1), *if A and B have no common factor.*

Any solution of (1), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0.

Then, it must make A identically equal to 0.

Then, it is a solution of (2).

Again, any solution of (2), when substituted for the unknown numbers, makes A identically equal to 0.

Since A and B have no common factor, B cannot be 0 when this solution is substituted for the unknown numbers.

Then, any solution of (2), when substituted for the unknown numbers, makes $\frac{A}{B}$ identically equal to 0, and is a solution of (1).

Therefore, (1) and (2) are equivalent, if A and B have no common factor.

If A and B have a common factor, (1) *and* (2) *are not equivalent;* consider, for example, the equations

$$\frac{x-1}{x^2-1} = 0, \text{ and } x - 1 = 0.$$

The second equation is satisfied by the value $x = 1$, which does not satisfy the first equation; then, the equations are not equivalent.

34. A fractional equation may be cleared of fractions by multiplying both members by *any* common multiple of the denominators; but in this way additional solutions are often introduced, and the resulting equation is not equivalent to the first.

Consider, for example, the equation

$$\frac{x^2}{x^2-1}+\frac{x}{x-1}=2.$$

If we solve by multiplying both members by x^2-1, the L. C. M. of x^2-1 and $x-1$, we find $x=-2$.

If, however, we multiply both members by $(x^2-1)(x-1)$, we have

$$x^3-x^2+x^3-x=2x^3-2x^2-2x+2, \text{ or } x^2+x-2=0.$$

The latter equation may be solved by using factors.

The factors of x^2+x-2 are $x+2$ and $x-1$.

Solving the equation $x+2=0$, $x=-2$.

Solving the equation $x-1=0$, $x=1$.

This gives the additional value $x=1$; and it is evident that this does not satisfy the given equation.

35. If both members of an equation be raised to the same positive integral power (§ 66), the resulting equation will have all the solutions of the given equation, and, in general, additional ones.

Consider, for example, the equation $x=3$.

Squaring both members, we have

$$x^2=9, \text{ or } x^2-9=0, \text{ or } (x+3)(x-3)=0.$$

The latter equation has the root 3, and, in addition, the root -3.

We will now consider the general case.

Let
$$A=B \qquad (1)$$
be an equation involving one or more unknown numbers.

Raising both members to the nth power, n being a positive integer, we have
$$A^n=B^n, \text{ or } A^n-B^n=0. \qquad (2)$$

Factoring the first number (§ 103, VII),

$$(A-B)(A^{n-1}+A^{n-2}B+\cdots+B^{n-1})=0. \qquad (3)$$

Now, equation (3) is satisfied when $A=B$.

Whence, equation (2) has all the solutions of (1).

But (3) is also satisfied when

$$A^{n-1} + A^{n-2}B + \cdots + B^{n-1} = 0;$$

so that (2) has also the solutions of this last equation, which, in general, do not satisfy (1).

EQUIVALENT SYSTEMS OF EQUATIONS

36. Two systems of equations, involving two or more unknown numbers, are said to be *equivalent* when every solution of the first system is a solution of the second, and every solution of the second is a solution of the first.

37. If

$$\begin{cases} A = 0, \\ B = 0, \end{cases}$$

are equations involving two or more unknown numbers, the system of equations

$$\begin{cases} A = 0, \\ mA + nB = 0, \end{cases}$$

where m and n are any numbers, and n not equal to zero, is equivalent to the first system.

For any solution of the first system, when substituted for the unknown numbers, makes $A = 0$ and $B = 0$.

It then makes $A = 0$ and $mA + nB = 0$.

Then, it is a solution of the second system.

Again, any solution of the second system, when substituted for the unknown numbers, makes $A = 0$ and $mA + nB = 0$.

It therefore makes $nB = 0$, or $B = 0$.

Since it makes $A = 0$ and $B = 0$, it is a solution of the first system.

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations.

Either m or n may be *negative*.

38. If either equation, in a system of two, be solved for one of the unknown numbers, and the value found be substituted for this unknown number in the other equation, the resulting system will be equivalent to the first.

Let

$$\begin{cases} A = B, & (1) \\ C = D, & (2) \end{cases}$$

be equations involving two unknown numbers, x and y.

Let E be the value of x obtained by solving (1).

Let $F = G$ be the equation obtained by substituting E for x in (2).

To prove the system of equations

$$\begin{cases} x = E, & (3) \\ F = G, & (4) \end{cases}$$

equivalent to the first system.

Any solution of the first system satisfies (3), for (3) is only a form of (1).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting E for x in (2).

Then, any solution of the first system satisfies (4).

Again, any solution of the second system satisfies (1), for (1) is only a form of (3).

Also, the values of x and y which form the solution make x and E equal; and hence satisfy the equation obtained by putting x for E in (4).

Then, any solution of the second system satisfies (2).

Hence, the systems are equivalent.

A similar result holds for a system of any number of equations, involving any number of unknown numbers.

39. The principles of §§ 27, 28, 29, 31, 33, 35, 36, and 37 hold for equations of any degree.

40. In the solution of an equation of Exercise 1, we replaced each equation by an equivalent one more easily solved for the unknown number.

41. Elimination is the process of deriving from a system of two or more equations, a system containing one less unknown number than the given system.

There are several methods of elimination, each method depending on a process which will form a second system equivalent to the first.

42. A system of equations is called **Simultaneous** when each contains two or more unknown numbers, and every equation of the system is satisfied by the same set, or sets, of values of the unknown numbers; thus, each equation of the system

$$\begin{cases} x + y = 6, \\ x - y = 3, \end{cases}$$

is satisfied by the set of values $x = 4$, $y = 1$.

A **Solution** of a system of simultaneous equations is a set of values of the unknown numbers which satisfies every equation of the system; to *solve* a system of simultaneous equations is to find its solutions.

Ex. Solve

$$\left.\begin{array}{ll}(1) & 2x+5y=9,\\ (2) & x-y=1,\end{array}\right\}\quad \text{I.}$$

$$\left.\begin{array}{ll}(1) & 2x+5y=9,\\ (3) & 5x-5y=5,\end{array}\right\}\quad \text{II.}$$

$$\left.\begin{array}{ll}(1) & 2x+5y=9,\\ (4) & (2x+5y)+5x-5y=9+5,\end{array}\right\},\ \text{or}\ \left.\begin{array}{r}2x+5y=9,\\ 7x=14,\end{array}\right\}\quad \text{III.}$$

System II is equivalent to system I, and system III is equivalent to system II.

System III gives the required solution since (4) gives $x=2$ and this value substituted in (1) gives $y=1$.

Similarly it may be shown that elimination by substitution and by comparison involve the deriving of equivalent systems from the given system (§§ 37, 38).

Unless the equations of a given system are independent a solution is not possible.

43. If two equations, containing two or more unknown numbers, are not equivalent, they are called **Independent.**

Consider the equations

$$\begin{cases} x+y=5, & (1)\\ x+y=6. & (2)\end{cases}$$

It is evidently impossible to find a set of values of x and y which shall satisfy both (1) and (2).

Such equations are called **Inconsistent.**

EXERCISE 4

Solve the following equations, using Addition or Subtraction, Substitution or Comparison:

1. $\begin{cases} 3x+5y=21.\\ 7x-2y=8.\end{cases}$

2. $\begin{cases} x-2y=9.\\ 2x-y=12.\end{cases}$

3. $\begin{cases} 4x - 3y = 1. \\ 6x + 15y = 8. \end{cases}$

4. $\begin{cases} 2x - y = -3. \\ 6x + 9 = 3y. \end{cases}$

5. $\begin{cases} y = 4 + x. \\ 3x - 3y = -12. \end{cases}$

6. $\begin{cases} \frac{2}{3}x + \frac{1}{2}y = 7. \\ 3x - \frac{1}{6}y = 17. \end{cases}$

7. $\begin{cases} 3x - 2y = 1\frac{1}{2}. \\ x = 2y. \end{cases}$

8. $\begin{cases} \frac{1}{2}m + \frac{2}{3}n = -2. \\ 3m + 12 = -4n. \end{cases}$

9. $\begin{cases} s + 4v = -1. \\ v = 2s - 16. \end{cases}$

10. $\begin{cases} \dfrac{x+2}{7} = \dfrac{y}{5} - 1. \\ \dfrac{2x-9}{3} + \dfrac{2y-5}{5} = 10. \end{cases}$

11. $\begin{cases} \dfrac{17p - q}{7} = p - 3q. \\ 8p + q = 15. \end{cases}$

12. $\begin{cases} 3x + \dfrac{x-y}{3} = 25. \\ 15 - 2x + \dfrac{y}{5} = 0. \end{cases}$

13. $\begin{cases} 11t = u + 19. \\ 2t - u = 10. \end{cases}$

14. $\begin{cases} \dfrac{2}{3}(x - 3y) - \dfrac{2x - y}{2} = -5. \\ \dfrac{2x + 3y - 6}{9} - \dfrac{x - y}{7} = 1. \end{cases}$

15. $\begin{cases} \dfrac{6x - 5y + 10}{11} - \dfrac{5x + 3y}{7} = \dfrac{4}{15}. \\ 5y - 3x - 1 = 0. \end{cases}$

16. $\begin{cases} y - 3x = a. \\ \frac{3}{2}x + \frac{6}{7}y = 9a. \end{cases}$

17. $\begin{cases} \dfrac{x + y + 2}{4} - \dfrac{3x - y}{17} - 5 = \dfrac{x}{6}. \\ \dfrac{x}{9} + \dfrac{y}{5} = 6. \end{cases}$

18. $\begin{cases} \dfrac{x + y - 1}{27} - \dfrac{1}{5}(x - y) = 1. \\ 4y = 17\frac{1}{2} + \frac{11}{4}x. \end{cases}$

19. $\begin{cases} 2\,ax - 4\,by = a^2 - ab + 2\,b^2. \\ x + y = a. \end{cases}$

20. $\begin{cases} \dfrac{x+y}{4} = c + d. \\ 2\,x - y = 5\,c - 7\,d. \end{cases}$

21. $\begin{cases} \dfrac{x}{17} + \dfrac{y}{15} = 10. \\ \dfrac{x}{5} - \dfrac{y}{12} = 10\frac{3}{4}. \end{cases}$

22. If 5 in. be added to the length and 3 in. to the breadth of a certain rectangle, the area is increased by 120 sq. in., but if 4 in. be subtracted from the length and 2 in. from the breadth, the area is decreased by 70 sq. in. Find its dimensions.

23. 2 cu. ft. of water and 4 cu. ft. of ice together weigh 355 lb. The difference between the weights of 3 cu. ft. of water and 2 cu. ft. of ice is 72 lb. 8 oz. Find the weights of a cubic foot of each.

24. A masonry contractor held back \$132.50 of the wages due his men. His bricklayers earned \$3 per day, and his hod carriers \$1.75 per day. Their combined wages for a day were \$256.25. He retained \$1.50 from each bricklayer and \$1 from each hod carrier. How many carriers did he employ?

25. A man rows a certain distance down stream at the rate of $3\frac{1}{2}$ mi. an hour in $3\frac{1}{2}$ hr. In returning it takes him 16 hr. to reach a point 5 mi. below his starting point. Find the rate of the current.

26. Two trains start toward each other, one from New York, the other from Chicago. They meet in 10 hr., 40 min., the distance between the two cities being 960 mi. If the first train starts 3 hr. earlier than the second train, they will meet $9\frac{1}{3}$ hr. after the second train starts. Find the rate of each train.

27. A number lies between 300 and 400. If 18 is added to the number, the last two digits change places with each other, and if the number be divided by the number expressed by the first two digits, the quotient is $10\frac{3}{17}$. Find the number.

28. Find two numbers whose difference is 93 and whose sum divided by the smaller number gives a quotient of $6\frac{3}{7}$.

29. By the law of levers, the product of the weight W_1 by the distance from W_1 to the fulcrum, F, is equal to the product of the weight W_2 by the distance from W_2 to the fulcrum.

A board resting across a pole balances when a 60-lb. boy is on one end and a 100-lb. boy on the other end. The board will also balance if a 120-lb. boy sits 2 ft. from one end and a 60-lb. boy sits 2 ft. from the other end. Find the length of the board.

30. If a regular hexagon is circumscribed about a given circle, the difference between the areas of the hexagon and circle is 32.24, and the sum of their areas is 660.56. Find the radius of the circle.

GRAPHICAL REPRESENTATION

44. A drawing or picture of given data or of an equation is often of value.

45. Descartes (1596–1650) was the first mathematician to apply measurement to equations.

It is impossible to locate absolutely a point in a plane. All measurements are purely relative, and all positions in a plane or in space are likewise relative. Since a plane is infinite in length and breadth, it is necessary to have some fixed form from which one can take measurements. For this form, assumed fixed in a plane, Descartes chose two intersecting lines as a coördinate system. Such a system of coördinates has since his time been called Cartesian. It will best suit our purpose to choose lines intersecting at right angles.

46. The Point. If we take any point M, its position is determined by the length of the lines $QM = x$ and $PM = y$, parallel to the intersecting lines OX and OY (Fig. 1). The values $x = a$ and $y = b$ will thus determine a point. The unit of length can be arbitrarily chosen, but when once fixed remains the same throughout the problem under discussion. $QM = x$ and $PM = y$, we call the *coördinates* of the point M. x, measured parallel to OX, is called the *abscissa.* y, measured parallel to OY, is the *ordinate.* OX and OY are the *coördinate axes.* OX is the axis of x, also called the axis of abscissas. OY is the axis of y, also called the axis of ordinates. O, the point of intersection, is called the *origin.*

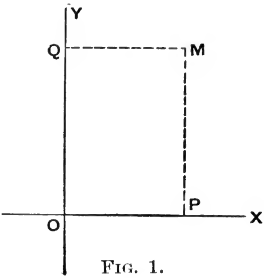


FIG. 1.

Two measurements are necessary to locate a point in a plane.

For example, $x = 2$ holds for any point on the line AB (Fig. 2). But if in addition we demand that $y = 3$, the point is fully determined by the intersection of the lines AB and CD, any point on CD satisfying the equation $y = 3$.

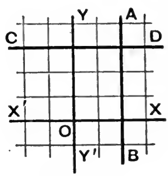


FIG. 2.

47. The Line. Consider the equation $x + y = 6$.

In this equation, when values are assigned to x, we get a value of y for every such value of x. When $x = 0$, $y = 6$; $x = 1$, $y = 5$; $x = 2$, $y = 4$; $x = 3$, $y = 3$; $x = 5$, $y = 1$; etc., giving an infinite number of values of x and y which satisfy the equation.

Laying off these values on a pair of axes, as shown in § 46, we see that the points whose coördinates satisfy this equation lie on the line AB (Fig. 3). It is readily seen that there might be confusion as to the direction from the origin in which the measurements should be taken. This is avoided by a simple convention in signs. Negative values of x are measured to the left of the y-axis, positive to the right. In like manner, negative values of y are measured downward from the x-axis, positive values upward. XOY, YOX', $X'OY'$, $Y'OX$, are spoken of as the first, second, third, and fourth quadrants respectively. (See Fig. 2.)

By plotting other equations of the *first degree* with two unknown quantities it will be seen that such an equation always represents a *straight line.* This line AB (Fig. 3) is called the graph of $x+y=6$ and is the locus of all the points satisfying that equation.

48. Now plot two simultaneous equations of the first degree on the same axes, *e.g.* $x+y=6$ and $2x-3y=-3$ (Fig. 4). We see that the coördinates of the point of intersection have the same values as the x and y of the algebraic solution of the equations.

This is a geometric or graphical reason why there is but one solution to a pair of simultaneous equations of the first degree with two unknown numbers. A simple algebraic proof will be given in the next article. Hereafter an equation of the first degree in two variables will be called a *linear equation.*

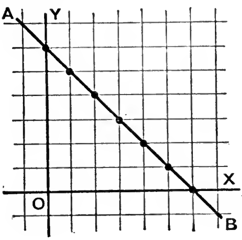


Fig. 3.

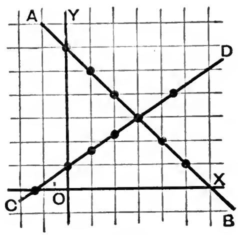


Fig. 4.

49. Algebraic Proof of the Principle of § 48. Two simultaneous equations of the first degree cannot be satisfied by two different sets of values for x and y. Given the equations

$$ax+by=c, \tag{1}$$

$$ex+fy=h. \tag{2}$$

Eliminating y, $$(af-eb)x=cf-bh. \tag{3}$$

Let x_1 and x_2 be the roots of (3), different in value. Substituting these roots, we have

$$(af-eb)x_1=cf-bh,$$

$$(af-eb)x_2=cf-bh.$$

$$(af-eb)(x_1-x_2)=0.$$

But $x_1 \neq x_2$, $\therefore af=eb$, or $\frac{a}{e}=\frac{b}{f}$, which is impossible.

In general, the plotting of two graphs on the same axes will determine all the *real* solutions of the two equations, the coördinates of each point of intersection of the graphs being values of x and y which satisfy both equations.

50. It is well to introduce the subject of graphs by the use of concrete problems which depend on two conditions and which can be solved without mention of the word *equation*.

Professor F. E. Nipher, Washington University, St. Louis, proposes the following:

"A person wishing a number of copies of a letter made, went to a typewriter and learned that the cost would be, for mimeograph work:

(1)
$1.00 for 100 copies,
$2.00 for 200 copies,
$3.00 for 300 copies,
$4.00 for 400 copies, and so on.

"He then went to a printer and was made the following terms:

(2)
$2.50 for 100 copies,
$3.00 for 200 copies,
$3.50 for 300 copies,
$4.00 for 400 copies, and so on, a rise of 50 cents for each hundred.

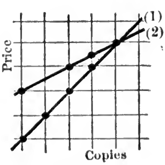


FIG. 5.

"Plotting the data of (1) and (2) on the same axes, we have:

"The vertical axis being chosen for the price-units, the horizontal axis for the number of copies.

"Any point on line (1) will determine the price for a certain number of mimeograph copies. Any point on line (2) determines the price and corresponding number of copies of printer's work."

Numerous lessons can be drawn from this problem. One is that for less than 400 copies, it is less expensive to patronize the mimeographer. For 400 copies, it does not matter which party is patronized. For *no* copies from the mimeographer, one pays nothing. How about the cost of *no* copies from the printer? Why?

The graph offers an excellent method for the solution of indeterminate equations in positive integers.

Ex. Solve $3x + 4y = 22$ for positive integers. Plotting the equation, we have

x	y
0	$5\frac{1}{2}$
2	4
3	$3\frac{1}{4}$
6	1

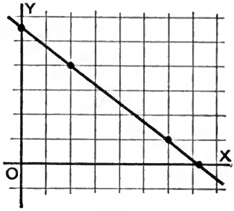


FIG. 6.

We see that the line crosses the corner of a square only when $x = 2$ and $x = 6$. For all other integral values of x, y is fractional. The only positive integral solutions are, therefore, $x = 2$, $y = 4$; $x = 6$, $y = 1$. This corresponds to the algebraic result.

51. In the equation $y = \frac{22 - 3x}{4}$, y is dependent on x for its value. That is, every change in x produces a change in y. When two quantities are so related, the first is said to be a **Function** of the second. Similarly $y = f(x)$, read y is a function of x, means that y is equal to some expression in x. In place of the equation represented by Fig. 6 one might have

$$f(x) = \frac{22 - 3x}{4}.$$

$$\begin{cases} f(x) = 8 - 2x, \\ F(x) = 4 + x.^* \end{cases}$$

Make a graph of each of these two functions and find their point of intersection.

EXERCISE 5

1. $f(x) = 7x - 24$, find $f(0)$, $f(1)$, $f(2)$, $f(-4)$, $f(3\frac{3}{7})$.

2. $\phi(x) = x^2 - 2x + 1$, find $\phi(0)$, $\phi(1)$, $\phi(2)$.

* The $f(x)$ and $F(x)$ mean simply different functions of x. In these same equations $f(0)$ means the value of the function when 0 is substituted for x in $f(x) = 8 - 2x$, namely, $f(0) = 8$. Similarly $f(1) = 8 - 2(1) = 6$.

Solve the following by means of graphs:

3. $\begin{cases} 2x - 5y = -16. \\ 3x + 7y = 5. \end{cases}$

4. $\begin{cases} \dfrac{x-5}{4} - \dfrac{2x-y-1}{3} = \dfrac{2y-2}{5}. \\ \dfrac{2y+x-1}{9} = \dfrac{x+y}{4}. \end{cases}$

5. $\begin{cases} f(y) = \dfrac{3y-62}{7}. \\ F(y) = \dfrac{2y-44}{5}. \end{cases}$

6. $\begin{cases} \phi(y) = \dfrac{1-8y}{15}. \\ \psi(y) = \dfrac{7y-24}{10}. \end{cases}$

7. $\begin{cases} f(x) = \dfrac{5x-19}{3}. \\ F(x) = \dfrac{2-7x}{4}. \end{cases}$

INEQUALITIES

52. The **Signs of Inequality,** $>$ and $<$, are read "*is greater than*" and "*is less than,*" respectively.

Thus, $a > b$ is read "a is greater than b"; $a < b$ is read "a is less than b."

53. One number is said to be *greater* than another when the remainder obtained by subtracting the second from the first is a *positive* number.

One number is said to be *less* than another when the remainder obtained by subtracting the second from the first is a *negative* number.

Thus, if $a - b$ is a positive number, $a > b$; and if $a - b$ is a negative number, $a < b$.

54. An **Inequality** is a statement that one of two expressions is greater or less than another.

The *First Member* of an inequality is the expression to the left of the sign of inequality, and the *Second Member* is the expression to the right of that sign.

Any term of either member of an inequality is called a *term* of the inequality.

55. Two or more inequalities are said to *subsist in the same sense* when the first member is the greater or the less in both.

Thus, $a > b$ and $c > d$ subsist in the same sense.

PROPERTIES OF INEQUALITIES

56. An inequality will continue in the same sense after the same number has been added to, or subtracted from, both members.

For consider the inequality $a > b$.

By § 53, $a - b$ is a positive number.

Hence, each of the numbers

$$(a + c) - (b + c), \text{ and } (a - c) - (b - c)$$

is positive, since each is equal to $a - b$.

Therefore, $a + c > b + c$, and $a - c > b - c$. (§ 53)

57. It follows from § 56 that **a term may be transposed from one member of an inequality to the other by changing its sign.**

If the same term appears in both members of an inequality, affected with the same sign, it may be removed.

58. If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.

For consider the inequality $a - b > c - d$.

Transposing every term, $d - c > b - a$. (§ 57)

That is, $b - a < d - c$.

59. An inequality will continue in the same sense after both members have been multiplied or divided by the same positive number.

For consider the inequality $a > b$.

By § 53, $a - b$ is a positive number.

Hence, if m is a positive number, each of the numbers

$$m(a - b) \text{ and } \frac{a - b}{m}, \text{ or } ma - mb \text{ and } \frac{a}{m} - \frac{b}{m}, \text{ is positive.}$$

Therefore, $ma > mb$, and $\frac{a}{m} > \frac{b}{m}$.

60. It follows from §§ 58 and 59 that **if both members of an inequality be multiplied or divided by the same negative number, the sign of inequality must be reversed.**

61. If any number of inequalities, subsisting in the same sense, be added member to member, the resulting inequality will also subsist in the same sense.

For consider the inequalities $a > b$, $a' > b'$, $a'' > b''$, $\cdots$.

Each of the numbers, $a - b$, $a' - b'$, $a'' - b''$, $\cdots$, is positive.

Then, their sum $a - b + a' - b' + a'' - b'' + \cdots$,

or
$$a + a' + a'' + \cdots - (b + b' + b'' + \cdots),$$
is a positive number.

Whence,
$$a + a' + a'' + \cdots > b + b' + b'' + \cdots.$$

If two inequalities, subsisting in the same sense, be *subtracted* member from member, the resulting inequality does not necessarily subsist in the same sense.

Thus, if $a > b$ and $a' > b'$, the numbers $a - b$ and $a' - b'$ are positive.

But $(a - b) - (a' - b')$, or its equal, $(a - a') - (b - b')$, may be positive, negative, or zero; and hence $a - a'$ may be greater than, less than, or equal to $b - b'$.

62. If $a > b$ and $a' > b'$, and each of the numbers a, a', b, b', is positive, then
$$aa' < bb'.$$

Since $a' > b'$, and a is positive,
$$aa' > ab' \text{ (§ 59).} \qquad (1)$$

Again, since $a > b$, and b' is positive,
$$ab' > bb'. \qquad (2)$$

From (1) and (2),
$$aa' > bb'.$$

63. If we have any number of inequalities subsisting in the same sense, as $a > b$, $a' > b'$, $a'' > b''$, $\cdots$, and each of the numbers a, a', a'', $\cdots$, b, b', b'', $\cdots$, is positive, then
$$aa'a'' \cdots > bb'b'' \cdots.$$

For by § 62, $\quad aa' > bb'.$

Also, $\quad a'' > b''.$

Then by § 62, $\quad aa'a'' > bb'b''.$

Continuing the process with the remaining inequalities, we obtain finally
$$aa'a'' \cdots > bb'b'' \cdots.$$

64. Examples.

1. Find the limit of x in the inequality
$$7x - \frac{23}{3} < \frac{2x}{3} + 5.$$

Multiplying both members by 3 (§ 59), we have

$$21x - 23 < 2x + 15.$$

Transposing (§ 57), and uniting terms,

$$19x < 38.$$

Dividing both members by 19 (§ 59),

$$x < 2.$$

(This means that, for any value of $x < 2$, $7x - \frac{23}{3} < \frac{2x}{3} + 5$.)

2. Find the limits of x and y in the following:

$$\begin{cases} 3x + 2y > 37. & (1) \\ 2x + 3y = 33. & (2) \end{cases}$$

Multiply (1) by 3,	$9x + 6y > 111.$
Multiply (2) by 2,	$4x + 6y = 66.$
Subtracting (§ 56),	$5x > 45$, and $x > 9$.
Multiply (1) by 2,	$6x + 4y > 74.$
Multiply (2) by 3,	$6x + 9y = 99.$
Subtracting,	$-5y > -25$

Divide both members by -5, $y < 5$ (§ 60).

(This means that any values of x and y which satisfy (2), also satisfy (1), provided x is > 9, and $y < 5$.)

3. Between what limiting values of x is $x^2 - 4x < 21$?

Transposing 21, we have

$$x^2 - 4x < 21, \text{ if } x^2 - 4x - 21 < 0.$$

That is, if $(x + 3)(x - 7)$ is negative.

Now $(x + 3)(x - 7)$ is negative if x is between -3 and 7; for if $x < -3$, both $x + 3$ and $x - 7$ are negative, and their product positive; and if $x > 7$, both $x + 3$ and $x - 7$ are positive.

Hence, $x^2 - 4x < 21$, if $x > -3$, and < 7.

EXERCISE 6

Find the limits of x in the following:

1. $(4x + 5)^2 - 4 < (8x + 5)(2x + 3)$.
2. $(3x + 2)(x + 3) - 4x > (3x - 2)(x - 3) + 36$.
3. $(x + 4)(5x - 2) + (2x - 3)^2 > (3x + 4)^2 - 78$.

4. $(x-3)(x+4)(x-5)<(x+1)(x-2)(x-3)$.

5. $a^2(x-1)<2b^2(2x-1)-ab$, if $a-2b$ is positive.

Find the limits of x and y in the following:

6. $\begin{cases} 5x+6y<45. \\ 3x-4y=-11. \end{cases}$

7. $\begin{cases} 7x-4y>41. \\ 3x+7y=35. \end{cases}$

8. Find the limits of x when

$$3x-11<24-11x, \text{ and } 5x+23<20x+3.$$

9. If 6 times a certain positive integer, plus 14, is greater than 13 times the integer, minus 63, and 17 times the integer, minus 23, is greater than 8 times the integer, plus 31, what is the integer?

10. If 7 times the number of houses in a certain village, plus 33, is less than 12 times the number, minus 82, and 9 times the number, minus 43, is less than 5 times the number, plus 61, how many houses are there?

11. A farmer has a number of cows such that 10 times their number, plus 3, is less than 4 times the number, plus 79; and 14 times their number, minus 97, is greater than 6 times the number, minus 5. How many cows has he?

12. Between what limiting values of x is $x^2+3x<4$?

13. Between what limiting values of x is $x^2<8x-15$?

14. Between what limiting values of x is $3x^2+19x<-20$?

65. If a and b are unequal numbers,

$$a^2+b^2>2ab.$$

For $(a-b)^2>0$; or, $a^2-2ab+b^2>0$.

Transposing $-2ab$, $a^2+b^2>2ab$.

1. Prove that, if a does not equal 3,

$$(a+2)(a-2)>6a-13.$$

By the above principle, if a does not equal 3,

$$a^2 + 9 > 6a.$$

Subtracting 13 from both members,

$$a^2 - 4 > 6a - 13, \text{ or } (a+2)(a-2) > 6a - 13.$$

2. Prove that, if a and b are unequal positive numbers,

$$a^3 + b^3 > a^2b + b^2a.$$

We have, $a^2 + b^2 > 2ab$, or $a^2 - ab + b^2 > ab$.

Multiplying both members by the positive number $a + b$,

$$a^3 + b^3 > a^2b + b^2a.$$

EXERCISE 7

1. Prove that for any value of x, except $\frac{5}{3}$,

$$3x(3x - 10) > -25.$$

2. Prove that for any value of x, except $\frac{7}{2}$,

$$4x(x - 5) > 8x - 49.$$

3. Prove that for any values of a and b, if $4a$ does not equal $3b$, $(4a + 3b)(4a - 3b) > 6b(4a - 3b)$.

4. Prove that for any values of x and y, if $5x$ does not equal $4y$, $5x(5x - 6y) > 2y(5x - 8y)$.

Prove that, if a and b are unequal positive numbers,

5. $a^3b + ab^3 > 2a^2b^2$.

6. $a^3 + a^2b + ab^2 + b^3 > 2ab(a + b)$.

III. EXPONENTS

66. An **Exponent** is a number written at the right of and above a number.

It is customary to speak of the number as raised to the power indicated by the exponent.

67. The laws we shall develop are to hold for any exponent, whether integral, fractional, positive, negative, or zero.

68. The number raised to the power is called the **Base**.

69. Meaning of a Positive Integral Exponent.

$$a^3 = a \cdot a \cdot a.$$

$$a^4 = a \cdot a \cdot a \cdot a.$$

Similarly if m is a positive integer,

$$a^m = a \cdot a \cdot a \cdot \cdots \text{ to } m \text{ factors.}$$

The following results have been proved to hold for any positive integral values of m and n:

$$a^m \times a^n = a^{m+n} \text{ (F. C.)*} \qquad (1)$$

$$(a^m)^n = a^{mn} \text{ (F. C.)} \qquad (2)$$

70. Meaning of a Fractional Exponent.

Let it be required to find the meaning of $a^{\frac{5}{3}}$.

If (1), § 69, is to hold for all values of m and n,

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Then, the *third power* of $a^{\frac{5}{3}}$ equals a^5.

Hence, $a^{\frac{5}{3}}$ must be the *cube root* of a^5, or $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

We will now consider the general case.

Let it be required to find the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

* F. C. refers to Wells's First Course in Algebra.

If (1), § 69, is to hold for all values of m and n,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Then, the qth power of $a^{\frac{p}{q}}$ equals a^p.

Hence, $a^{\frac{p}{q}}$ must be the qth root of a^p, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, in a fractional exponent, the numerator denotes a power, and the denominator a root.

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

A **Surd** is the indicated root of a number, or expression, which is not a perfect power of the degree denoted by the index of the radical sign; as $\sqrt{2}$, $\sqrt[3]{5}$, or $\sqrt[4]{x+y}$.

The **degree** of a surd is denoted by its index; thus, $\sqrt[3]{5}$ is a surd of the third degree.

A **quadratic surd** is a surd of the second degree.

71. Meaning of a Zero Exponent.

If (1), § 69, is to hold for all values of m and n, we have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence, $$a^0 = \frac{a^m}{a^m} = 1.$$

We must then define a^0 as being equal to 1.

72. Meaning of a Negative Exponent.

Let it be required to find the meaning of a^{-3}.

If (1), § 69, is to hold for all values of m and n,

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1 \text{ (§ 71)}.$$

Whence, $$a^{-3} = \frac{1}{a^3}.$$

We will now consider the general case.

Let it be required to find the meaning of a^{-s}, where s represents a positive integer or a positive fraction.

If (1), § 69, is to hold for all values of m and n,

$$a^{-s} \times a^{s} = a^{-s+s} = a^{0} = 1 \text{ (§ 71).}$$

Whence, $$a^{-s} = \frac{1}{a^{s}}.$$

We must then define a^{-s} as being equal to 1 divided by a^{s}.

For example, $a^{-2} = \frac{1}{a^{2}}$; $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$; $3x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}}$; etc.,

73. It follows from § 72 that

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $$\frac{a^2b^3}{cd^4} = \frac{b^3}{a^{-2}cd^4} = \frac{a^2b^3c^{-1}}{d^4} = \frac{a^2d^{-4}}{b^{-3}c}, \text{ etc.}$$

EXERCISE 8

Express with positive exponents:

1. $a^{-2}b^{3}$.
2. $x^{\frac{3}{4}}y^{-2}z$.
3. $2m^{-4}n$.
4. $a^{-1}b^{4}c^{-3}$.
5. $3xyz^{-2}$.
6. $5c^{-\frac{1}{2}}d^{\frac{1}{3}}$.
7. $a^{-2}xy^{-5}$.
8. $3p^{-1}q^{\frac{1}{2}}$.
9. $7x^{4}y^{-2}z$.
10. $4a^{-6}b^{-8}c^{\frac{3}{2}}$.
11. $8u^{3}v^{-1}$.
12. $r^{\frac{1}{4}}s^{-\frac{1}{2}}t^{-\frac{1}{3}}$.

Transfer all literal factors from the denominators to the numerators:

13. $\frac{6x^3}{y}$.
14. $\frac{mn^{-4}}{3x^2}$.
15. $\frac{abc^{-1}}{xy^2z}$.
16. $\frac{1}{2a^2b^{-3}}$.
17. $\frac{a^4}{cd^{-3}}$.
18. $\frac{3x^2y^{-\frac{1}{4}}}{4z^4}$.
19. $\frac{5a^{\frac{1}{2}}b^{-\frac{1}{3}}}{9c^{-3}d^{\frac{3}{4}}}$.
20. $\frac{4m^{-2}n^3}{7x^{-8}y^{-\frac{5}{2}}z}$.

Transfer all literal factors from the numerators to the denominators:

21. $\dfrac{a^3b^2}{x^4}$.

22. $\dfrac{7x^2y^{-1}}{m^3n^{\frac{3}{2}}}$.

23. $\dfrac{3ab^2c^{-1}}{5d^{-4}}$.

24. $\dfrac{p^{-8}q^{\frac{3}{5}}}{5x^3}$.

25. $\dfrac{m^6}{n^{-7}r^{\frac{3}{2}}}$.

26. $\dfrac{8x^{-4}yz^3}{3cd^2}$.

27. $\dfrac{a^{-2}}{b^2c}$.

28. $\dfrac{3m^{\frac{1}{4}}n^{-\frac{2}{3}}}{4xy^{-1}z^2}$.

74. Proof that $a^m \cdot a^n = a^{m+n}$ holds for all values of m and n.

I. Let $m = \dfrac{p}{q}$ and $n = \dfrac{r}{s}$, where p, q, r, and s are positive integers.

We have, $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}} = \sqrt[qs]{a^{ps}} \times \sqrt[qs]{a^{qr}}$ (§ 70)

$$= \sqrt[qs]{a^{ps} \times a^{qr}} = \sqrt[qs]{a^{ps+qr}} = a^{\frac{ps+qr}{qs}} \text{ (§ 70)} = a^{\frac{p}{q}+\frac{r}{s}}.$$

We have now proved that (1), § 69, holds when m and n are any positive integers or positive fractions.

II. Let m be a positive integer or fraction; and let $n = -q$, where q is a positive integer or fraction less than m.

By § 74, I, $a^{m-q} \times a^q = a^{m-q+q} = a^m$.

Whence, $a^{m-q} = \dfrac{a^m}{a^q} = a^m \times a^{-q}$ (§ 73).

That is, $a^m \times a^{-q} = a^{m-q}$.

III. Let m be a positive integer or fraction; and let $n = q$, where q is a positive integer or fraction greater than m.

By § 73, $a^m \times a^{-q} = \dfrac{1}{a^{-m}a^q} = \dfrac{1}{a^{-m+q}}$ (§ 74, II) $= a^{m-q}$.

IV. Let $m = -p$ and $n = -q$, where p and q are positive integers or fractions.

Then, $a^{-p} \times a^{-q} = \dfrac{1}{a^p a^q} = \dfrac{1}{a^{p+q}}$ (74, I) $= a^{-p-q}$.

Then, $a^m \times a^n = a^{m+n}$ for all positive or negative, integral or fractional, values of m and n.

EXERCISE 9

Multiply the following:

1. a^8 by $a^{-\frac{3}{2}}$.

2. $3x^{\frac{1}{2}}y^{-\frac{1}{3}}z$ by $x^{-\frac{3}{4}}yz^3$.

3. $2c^{\frac{1}{4}}d$ by $3\sqrt{cd^{\frac{1}{2}}}$.

4. $2\sqrt[4]{ab^{-3}}$ by $\sqrt[3]{a^2b}$.

5. $x^{\frac{3}{4}}y$ by $\dfrac{1}{x^{-\frac{5}{6}}y^{-2}}$.

6. a^4bc^2 by $ab^{-1}c^{-2}$.

7. $3x^{-4}y^0$ by $-2x^6y^3z$.

8. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

9. $3x-1+x^{-1}$ by $5x+2$.

10. $x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $\sqrt{x}+\sqrt{y}$.

11. $x^2+2x-x^{-1}+1$ by $x+x^{-1}+1$.

12. $\dfrac{1}{x^{-2}}-x^{\frac{3}{2}}+x-\sqrt{x}+1$ by $\sqrt{x}+1$.

Divide the following:

13. a^2 by a^5.

14. $x^{\frac{1}{4}}y$ by $x^{-3}y^2$.

15. $3\sqrt{xy^3}$ by $x^{-\frac{1}{5}}y^{-1}$.

16. $8a^{\frac{1}{2}}b^{-\frac{1}{3}}\sqrt[3]{c^2}$ by $2a^{-2}bc^3$.

17. $5m^4n^{-\frac{1}{6}}$ by $2m^{-\frac{5}{2}}n^{\frac{1}{3}}$.

18. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

19. $m^{-\frac{3}{2}}+3m^{-1}n^{\frac{1}{2}}+3m^{-\frac{1}{2}}n+n^{\frac{3}{2}}$ by $m^{-\frac{1}{2}}+n^{\frac{1}{2}}$.

20. $6x^2-6x^{-2}-12-11x+23x^{-1}$ by $2x+1-3x^{-1}$.

21. $9x^{\frac{1}{3}}y^{-1}-6x^{\frac{2}{3}}y^{-2}-5+12x^{-\frac{1}{3}}y+4x^{-\frac{2}{3}}y^2$ by $3x+4\sqrt[3]{xy^2}$.

22. $2x^{\frac{3}{4}}-7x^{\frac{1}{2}}+4x^{-\frac{1}{4}}-11+12x^{\frac{1}{4}}$ by $2x^{\frac{1}{2}}+4-3x^{\frac{1}{4}}$.

Find the values of the following:

23. $(x^2)^{-3}$.

24. $(m^{-2})^{-\frac{3}{4}}$.

25. $(\sqrt[3]{x^{-2}})^{-\frac{5}{6}}$.

26. $(36)^{\frac{3}{2}}$.

27. $(-27)^{\frac{5}{3}}$.

28. $(8^{-\frac{1}{3}})^4$.

29. $\left(x^{\frac{m}{m+n}}\right)^{1-\frac{n^2}{m^2}}$.

30. $(\sqrt{a^q})^{-\frac{1}{q}}$.

75. To prove the result

$$(ab)^n = a^n b^n,$$

for any fractional or negative value of n.

The proof of this result in the case where n is any positive integer, was given in F. C.

I. Let $n = \frac{p}{q}$, where p and q are any positive integers.

$$[(ab)^{\frac{p}{q}}]^q = (ab)^p = a^p b^p \text{ (F. C.)}. \qquad (1)$$

$$(a^{\frac{p}{q}} b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q (b^{\frac{p}{q}})^q = a^p b^p. \qquad (2)$$

From (1) and (2), $[(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}} b^{\frac{p}{q}})^q$.

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

II. Let $n = -s$, where s is any positive integer or positive fraction.

Then, $(ab)^{-s} = \dfrac{1}{(ab)^s} = \dfrac{1}{a^s b^s}$ (§ 74, IV) $= a^{-s} b^{-s}$.

EXERCISE 10

Find the values of the following:

1. $(a^3 b^{\frac{1}{2}})^4$.
2. $\left(\dfrac{1}{x^{\frac{1}{2}} y^{-3}}\right)^{-\frac{2}{3}}$.
3. $(p^{-\frac{2}{3}} q^{\frac{3}{4}})^{-\frac{5}{6}}$.
4. $(a^{-4} \sqrt{b^3} c^{-1})^{-\frac{1}{2}}$.
5. $(\sqrt[3]{x^4 y^{-3}})^{\frac{7}{8}}$.
6. $(25\, a^4)^{-\frac{5}{2}}$.
7. $(32\, a^{\frac{1}{4}} \sqrt[3]{b^{-2}})^{\frac{3}{5}}$.
8. $(343\, b^4 c^{-1} d^{\frac{1}{2}})^{-\frac{2}{3}}$.
9. $\left(\dfrac{9\, x y^{-2} z^{\frac{1}{3}}}{16\, m^{-4}}\right)^{-\frac{3}{2}}$.
10. $\left(\dfrac{3\, a^{-4} b^{\frac{1}{3}} c}{4\, x^{-\frac{2}{9}} y^{-\frac{1}{5}}}\right)^{-2}$.

EXERCISE 11

Illustrative Examples.

Ex. 1. Reduce $\sqrt{\frac{7}{8}}$ to its simplest form.

A surd is said to be in its *simplest form* when the expression under the radical sign is rational and integral, is not a perfect power of the degree denoted by any factor of the index of the surd, and has no factor which is a perfect power of the same degree as the surd.

$8 = 2^3$. To be a perfect square the exponents of the factors of the denominator must be even numbers. Hence multiplying both terms of the fraction by 2, we have,

$$\sqrt{\frac{7}{8}} = \sqrt{\frac{14}{16}} = \frac{\sqrt{14}}{4}.$$

Ex. 2. Reduce $\sqrt[4]{25}$ to its simplest form.

$$\sqrt[4]{25} = \sqrt{\sqrt{25}} = \sqrt{5}.$$

Ex. 3. Express $5\sqrt{7}$ entirely under the radical sign.

$$5\sqrt{7} = \sqrt{5^2}\sqrt{7}, \text{ or } (5^2)^{\frac{1}{2}}(7)^{\frac{1}{2}}.$$

By § 75, $\quad (5^2)^{\frac{1}{2}}(7)^{\frac{1}{2}} = (5^2 \cdot 7)^{\frac{1}{2}} = \sqrt{175}.$

Ex. 4. Reduce $(5)^{\frac{1}{3}}$, $\sqrt[4]{3}$ to the same degree.

The L. C. M. of the indices of the roots is 12. Hence,

$$5^{\frac{1}{3}} = 5^{\frac{4}{12}}, \quad 3^{\frac{1}{4}} = 3^{\frac{3}{12}}.$$

The surds are now of the twelfth degree.

Ex. 5. Find the product of $\sqrt{45}$ and $\sqrt{72}$.

$$(45)^{\frac{1}{2}}(72)^{\frac{1}{2}} = (3^2 \cdot 5)^{\frac{1}{2}} \cdot (3^2 \cdot 2^3)^{\frac{1}{2}} = (2^2 \cdot 3^4 \cdot 5 \cdot 2)^{\frac{1}{2}}$$
$$= 2 \cdot 3^2 (5 \cdot 2)^{\frac{1}{2}} = 18\sqrt{10}.$$

Reduce the following to their simplest form:

1. $\sqrt[4]{9}$.
2. $(27\,a^3)^{\frac{1}{6}}$.
3. $(45)^{\frac{1}{2}}$.
4. $(72)^{\frac{1}{2}}$.
5. $\sqrt[3]{128\,a^4b^2}$.
6. $3\sqrt{250\,a^2x}$.
7. $5(32\,a^5x^4y^3)^{\frac{1}{4}}$.
8. $7(80)^{\frac{1}{2}}$.
9. $(686)^{\frac{1}{3}}$.

10. $4\sqrt[5]{486}$.

11. $\sqrt{4x^2 - 5x^3y}$.

12. $(a^3 - 2a^2b + ab^2)^{\frac{1}{2}}$.

13. $(x-y)(ax^3m^5)^{\frac{1}{2}}$.

14. $(4x^2 - 24xb + 36b^2)^{\frac{1}{2}}$.

15. $\sqrt{(x^2+3x+2)(x^2+6x+8)}$.

16. $(\frac{1}{3})^{\frac{1}{2}}$.

17. $(\frac{3}{8})^{\frac{1}{2}}$.

18. $(\frac{1}{4})^{\frac{1}{3}}$.

19. $\frac{1}{2}(\frac{1}{5})^{\frac{1}{2}}$.

20. $2\sqrt{\frac{5}{72}}$.

21. $\left(\frac{8a^2}{75}\right)^{\frac{1}{2}}$.

22. $\frac{1}{m}\left(\frac{18m^4}{25n^2}\right)^{\frac{1}{3}}$.

23. $(x-a)\left(\frac{1}{x+a}\right)^{\frac{1}{2}}$.

24. $\sqrt{\frac{3p}{p-2q}}$.

25. $\frac{1}{(a+b)^2}\left(\frac{a^2-b^2}{6}\right)^{\frac{1}{2}}$.

Express entirely under the radical sign:

26. $2\sqrt{5}$.

27. $3(2)^{\frac{1}{3}}$.

28. $a(bc^2)^{\frac{1}{4}}$.

29. $5x\sqrt{3xy}$.

30. $(a+3b)\left(\frac{1}{a+3b}\right)^{\frac{1}{2}}$.

31. $(x+y)\left(\frac{x}{x^2-y^2}\right)^{\frac{1}{3}}$.

32. $\frac{1}{m-n}\sqrt{m^2+mn-2n^2}$.

33. $\frac{c+4}{c-1}\left(\frac{c^2+5c-6}{c^2+8c+16}\right)^{\frac{1}{2}}$.

Reduce the following to equivalent surds of the same degree:

34. $\sqrt{3}$, $\sqrt[3]{4}$.

35. $(2)^{\frac{1}{2}}$, $(\frac{1}{4})^{\frac{1}{3}}$, $(5)^{\frac{1}{4}}$.

36. $(x^2y)^{\frac{1}{3}}$, $(xy)^{\frac{1}{2}}$, $(x^3y^2)^{\frac{1}{4}}$.

37. $5\sqrt[3]{x}$, $3\sqrt{xy}$.

38. $(a-x)^{\frac{1}{2}}$, $(a+x)^{\frac{1}{3}}$, $(a^2+x^2)^{\frac{1}{6}}$.

39. $\sqrt[6]{x^3+y^3}$, $\sqrt[9]{x^4-y^4}$.

Simplify the following:

40. $(18)^{\frac{1}{2}} + 3(50)^{\frac{1}{2}} - 2(72)^{\frac{1}{2}}$.

41. $2(27)^{\frac{1}{2}} - 5(48)^{\frac{1}{2}} + 11(75)^{\frac{1}{2}}$.

42. $\sqrt[3]{54} + \sqrt[3]{250} - 2\sqrt[3]{128}$.

43. $\frac{5}{2}(12)^{\frac{1}{2}} - \frac{3}{4}(\frac{1}{3})^{\frac{1}{2}} + (3)^{\frac{1}{2}}$.

44. $8\sqrt[4]{80} - 2\sqrt[4]{405} + 18\sqrt[4]{\frac{5}{16}}$.

45. $(24\,a^3x)^{\frac{1}{2}} + 2(54\,a^3x^3)^{\frac{1}{2}} - 5(6\,a^5x)^{\frac{1}{2}}$.

46. $\left(\dfrac{a^3m^5}{c^4n}\right)^{\frac{1}{2}} - \left(\dfrac{amn^3}{c^2d^2}\right)^{\frac{1}{2}} + \left(\dfrac{a^5n^3}{m^3x^4}\right)^{\frac{1}{2}}$.

47. $(\frac{7}{2})^{\frac{1}{2}} + 3(\frac{1}{14})^{\frac{1}{2}} + \frac{1}{2}\sqrt{56}$.

48. $\sqrt[3]{2\,ax} - 3(4\,a^2x^2)^{\frac{1}{6}} + 5\sqrt[9]{8\,a^3x^3}$.

49. $\dfrac{1}{(x+y)^2}\sqrt{x^3 + 2\,x^2y + xy^2} + \dfrac{x}{x^2-y^2}\left(\dfrac{(x+y)^4}{x}\right)^{\frac{1}{2}}$.

Multiply the following:

50. $\sqrt{90}$ by $\sqrt{63}$.

51. $(35)^{\frac{1}{2}}$ by $(105)^{\frac{1}{2}}$.

52. $\sqrt[3]{54}$ by $\sqrt[3]{6}$.

53. $(3)^{\frac{1}{2}}$ by $(2)^{\frac{1}{3}}$.

54. $\sqrt[3]{2}$ by $\sqrt[4]{4}$.

55. $(\frac{1}{2})^{\frac{1}{2}}$ by $(\frac{3}{4})^{\frac{1}{2}}$ by $(\frac{2}{3})^{\frac{1}{2}}$.

56. $\sqrt{a^3xy}$ by $\sqrt[3]{ax^2y^2}$.

57. $2(5)^{\frac{1}{2}}$ by $3(15)^{\frac{1}{2}}$.

58. $5\sqrt[3]{40}$ by $6(5)^{\frac{1}{2}}$.

59. $\left(\dfrac{3\,a}{8\,b}\right)^{\frac{1}{2}} \cdot \left(\dfrac{7\,a^2}{27\,b}\right)^{\frac{1}{2}} \cdot \left(\dfrac{2\,b^3}{15\,a}\right)^{\frac{1}{2}}$.

60. $\sqrt{3\,ax} \cdot (2\,a^2)^{\frac{1}{3}} \cdot (6\,x^3)^{\frac{1}{4}}$.

61. $3(2)^{\frac{1}{2}} - 5(3)^{\frac{1}{2}}$ by $4(2)^{\frac{1}{2}} + 3(3)^{\frac{1}{2}}$.

62. $5\sqrt{7} + 6\sqrt{2}$ by $\sqrt{7} - 4\sqrt{2}$.

63. $2(8\,x)^{\frac{1}{2}} - 9(2\,y)^{\frac{1}{2}}$ by $(2\,x)^{\frac{1}{2}} - 3(2\,y)^{\frac{1}{2}}$.

64. $3(a-1)^{\frac{1}{2}} + 4(2\,a+5)^{\frac{1}{2}}$ by $2(a-1)^{\frac{1}{2}} - 10(2\,a+5)^{\frac{1}{2}}$.

65. $5\sqrt{\frac{3}{5}} - 2\sqrt{\frac{1}{3}}$ by $4\sqrt{\frac{3}{5}} + 9\sqrt{\frac{1}{3}}$.

Divide the following:

66. $\sqrt{72}$ by $\sqrt{6}$.

67. $2\sqrt{125}$ by $4\sqrt{5}$.

68. $(192)^{\frac{1}{3}}$ by $(12)^{\frac{1}{3}}$.

69. $(512)^{\frac{1}{3}}$ by $(16)^{\frac{1}{2}}$.

70. $(8\,a^3)^{\frac{1}{2}}$ by $(16\,a^4)^{\frac{1}{3}}$.

71. $\sqrt{722\,x^5}$ by $\sqrt{2\,a^4x}$.

72. $(\frac{1}{2})^{\frac{1}{3}}$ by $(\frac{1}{2})^{\frac{1}{2}}$.

73. $8\sqrt{a^3x^2}$ by $6\sqrt[3]{ax^2}$.

74. $(1\frac{1}{3})^{\frac{1}{2}}$ by $(9)^{\frac{1}{3}}$.

Find values of the following:

75. $(a\sqrt{5})^3$.

76. $[\frac{1}{2}(4)^{\frac{1}{3}}]^2$.

77. $(3\sqrt[5]{2a^2b})^4$.

78. $[(3)^{\frac{1}{3}}]^{\frac{1}{2}}$.

79. $[(3a)^{\frac{1}{2}}]^{\frac{1}{3}}$.

80. $\sqrt{\sqrt[3]{a^2-2ab+b^2}}$.

81. $[3+2(5)^{\frac{1}{2}}]^2$.

82. $(4\sqrt{5}-2\sqrt{7})^2$.

83. $[3(3)^{\frac{1}{2}}-5(2)^{\frac{1}{2}}][3(3)^{\frac{1}{2}}+3(2)^{\frac{1}{2}}]$.

84. $(7\sqrt{11}-5\sqrt{3})(7\sqrt{11}+5\sqrt{3})$.

Express each of the following with a rational denominator:

85. $\dfrac{1}{\sqrt{2}}$.

86. $\dfrac{2a}{\sqrt{5}}$.

87. $\dfrac{x}{(x)^{\frac{1}{2}}+(a)^{\frac{1}{2}}}$.

88. $\dfrac{a+b^{\frac{1}{2}}}{a-b^{\frac{1}{2}}}$.

89. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

90. $\dfrac{3(4)^{\frac{1}{2}}-2(3)^{\frac{1}{2}}}{2(3)^{\frac{1}{2}}+(10)^{\frac{1}{2}}}$.

91. $\dfrac{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}}$.

92. $\dfrac{(x^2+y^2)^{\frac{1}{2}}+(x^2-y^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}-(x^2-y^2)^{\frac{1}{2}}}$.

LOGARITHMS

76. Any number may be expressed as a power of some number chosen as base.

For example, $4=2^2$, $8=2^3$, $64=2^6$, etc. Numbers between 4 and 8 would be expressed by 2^n where n is 2 plus some fractional number. In such a case the exponent is called the **Logarithm of the Number to the Base 2.**

E.g. 2 is the logarithm of 4 to the base 2; 3 is the logarithm of 8 to the base 2, etc.

77. The **Common System** of logarithms has 10 for its base.

Every positive arithmetical number may be expressed, exactly or approximately, as a power of 10.

Thus, $100=10^2$; $13=10^{1.1139\cdots}$; etc.

When thus expressed, the corresponding exponent is called its **Logarithm to the Base 10**.

Thus, 2 is the logarithm of 100 to the base 10; a relation which is written $\log_{10} 100 = 2$, or simply $\log 100 = 2$.

Logarithms of numbers to the base 10 are called *Common Logarithms*, and, collectively, form the *Common System*.

They are the only ones used for numerical computations.

78. Any positive number, except unity, may be taken as the base of a system of logarithms; thus, if $a^x = m$, where a and m are positive numbers, then $x = \log_a m$.

A negative number is not considered as having a logarithm.

79. By §§ 71 and 72,

$$10^0 = 1, \qquad 10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10, \qquad 10^{-2} = \frac{1}{10^2} = .01,$$

$$10^2 = 100, \qquad 10^{-3} = \frac{1}{10^3} = .001, \text{ etc.}$$

Whence, by the definition of § 76,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$

$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$

$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

The second form for log .1, log .01, etc., is preferable in practice.

If no base is expressed, the base 10 is understood.

80. It is evident from § 79 that the common logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 negative.

81. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

Here, the characteristic is 1, and the mantissa .1139.

A negative logarithm is always expressed with a positive mantissa, which is done by adding and subtracting 10.

Thus, the negative logarithm -2.5863 is written $7.4137-10$. In this case, $7-10$ is the characteristic.

The negative logarithm $7.4137-10$ is sometimes written $\bar{3}.4137$; the negative sign over the characteristic showing that it alone is negative, the mantissa being always positive.

For reasons which will appear, only the mantissa of the logarithm is given in a table of logarithms of number; the characteristic must be found by aid of the rules of §§ 82 and 83.

82. It is evident from § 79 that the logarithm of a number between

1 and 10 is equal to $0 +$ a decimal;
10 and 100 is equal to $1 +$ a decimal;
100 and 1000 is equal to $2 +$ a decimal; etc.

Therefore, the characteristic of the logarithm of a number with *one* place to the left of the decimal point is 0; with *two* places to the left of the decimal point is 1; with *three* places to the left of the decimal point is 2; etc.

Hence, **the characteristic of the logarithm of a number greater than 1 is 1 less than the number of places to the left of the decimal point.**

For example, the characteristic of log 906328.51 is 5.

83. In like manner, the logarithm of a number between

1 and .1 is equal to $9 +$ a decimal -10;
.1 and .01 is equal to $8 +$ a decimal -10;
.01 and .001 is equal to $7 +$ a decimal -10; etc.

Therefore, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure is 9, with -10 after the mantissa; of a decimal with *one* cipher between its point and first significant figure is 8, with -10 after the mantissa; of a decimal with *two* ciphers between its point and first significant figure is 7, with -10 after the mantissa; etc.

Hence, to find the characteristic of the logarithm of a number less than 1, subtract the number of ciphers between the decimal point and first significant figure from 9, writing -10 after the mantissa.

For example, the characteristic of log .007023 is 7, with -10 written after the mantissa.

PROPERTIES OF LOGARITHMS

84. *In any system, the logarithm of* 1 *is* 0.

For by § 71, $a^0 = 1$; whence, by § 78, $\log_a 1 = 0$.

85. *In any system the logarithm of the base is* 1.

For, $a^1 = a$; whence, $\log_a a = 1$.

86. *In any system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.*

For if a is greater than 1, $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$. (The discussion of this form will be found in § 127.)

Whence, by § 78, $\log_a 0 = -\infty$.

No literal meaning can be attached to such a result as $\log_a 0 = -\infty$; it must be interpreted as follows:

If, in any system whose base is greater than unity, a number approaches the limit 0, its logarithm is negative, and increases indefinitely in absolute value.

87. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence, by § 78, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Multiplying the assumed equations,

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y = \log_a m + \log_a n.$

* ∞ stands for a number greater than any assigned number. See § 126.

In like manner, the theorem may be proved for the product of three or more factors.

By aid of § 87, the logarithm of a composite number may be found when the logarithms of its factors are known.

Ex. Given $\log 2 = .3010$, and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned}\log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3)\\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3\\ &= 3 \times \log 2 + 2 \times \log 3 = .9030 + .9542 = 1.8572.\end{aligned}$$

EXERCISE 12

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, $\log 7 = .8451$, find:

1. $\log 15$.
2. $\log 98$.
3. $\log 84$.
4. $\log 125$.
5. $\log 315$.
6. $\log 392$.
7. $\log 567$.
8. $\log 1225$.
9. $\log 1372$.
10. $\log 1875$.
11. $\log 2646$.
12. $\log 24696$.

88. In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Assume the equations

$$\left.\begin{array}{l}a^x = m\\ a^y = n\end{array}\right\}; \text{ whence, } \left\{\begin{array}{l}x = \log_a m,\\ y = \log_a n.\end{array}\right.$$

Dividing the assumed equations,

$$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}.$$

Whence, $$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

Ex. Given $\log 2 = .3010$; find $\log 5$.

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - .3010 = .6990$$

EXERCISE 13

Given log 2 = .3010, log 3 = .4771, log 7 = .8451, find:

1. $\log \frac{10}{7}$.
2. $\log \frac{27}{2}$.
3. $\log 11\frac{1}{9}$.
4. $\log 245$.
5. $\log 85\frac{3}{4}$.
6. $\log 175$.
7. $\log \frac{49}{48}$.
8. $\log 375$.
9. $\log \frac{54}{25}$.
10. $\log \frac{300}{49}$.
11. $\log 46\frac{2}{7}$.
12. $\log 2\frac{11}{35}$.

89. In any system, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Assume the equation $a^x = m$; whence, $x = \log_a m$.

Raising both members of the assumed equation to the pth power, $a^{px} = m^p$; whence, $\log_a m^p = px = p \log_a m$.

90. In any system, the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

For, $\log_a \sqrt[r]{m} = \log_a(m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$(§ 89).

91. Examples.

1. Given $\log 2 = .3010$; find $\log 2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017.$$

To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find $\log \sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596.$$

3. Given $\log 2 = .3010$, $\log 3 = .4771$, find $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By § 87, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \frac{1}{3} \log 2 + \frac{5}{4} \log 3 = .1003 + .5964 = .6967.$$

EXERCISE 14

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log 2^8$.
2. $\log 5^7$.
3. $\log 3^{\frac{4}{3}}$.
4. $\log 7^{\frac{2}{5}}$.
5. $\log 42^6$.
6. $\log 45^{\frac{1}{4}}$.
7. $\log 63^{\frac{1}{6}}$.
8. $\log 98^{\frac{7}{2}}$.
9. $\log 50^{\frac{3}{7}}$.
10. $\log \sqrt[5]{3}$.
11. $\log \sqrt[8]{5}$.
12. $\log \sqrt[13]{7}$.
13. $\log \sqrt[7]{8}$.
14. $\log \sqrt[4]{54}$.
15. $\log \sqrt[6]{225}$.
16. $\log \sqrt[9]{162}$.
17. $\log \sqrt[12]{\frac{7}{3}}$.
18. $\log \left(\frac{5}{2}\right)^{\frac{1}{5}}$.
19. $\log (3^{\frac{3}{2}} \times 100^{\frac{1}{3}})$.
20. $\log (5\sqrt[11]{3})$.
21. $\log \dfrac{\sqrt[7]{7}}{\sqrt[3]{2}}$.
22. $\log \dfrac{2^{\frac{2}{5}}}{5^{\frac{1}{6}}}$.
23. $\log \dfrac{\sqrt[4]{35}}{7^{\frac{3}{8}}}$.
24. $\log \dfrac{3^{\frac{4}{7}}}{\sqrt[9]{75}}$.

92. In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.

Suppose, for example, that $\log 3.053 = .4847$.

$$\text{Then, } \log 305.3 = \log(100 \times 3.053) = \log 100 + \log 3.053$$
$$= 2 + .4847 = 2.4847;$$
$$\log .03053 = \log(.01 \times 3.053) = \log .01 + \log 3.053$$
$$= 8 - 10 + .4847 = 8.4847 - 10; \text{ etc.}$$

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

For this reason, only mantissæ are given, in a table of Common Logarithms; for to find the logarithm of any number, we have only to find the mantissæ corresponding to its sequence of figures, and then prefix the characteristic in accordance with the rules of §§ 82 and 83.

This property of logarithms only holds for the common system, and constitutes its superiority over other systems for numerical computation.

93. *Ex.* Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

We have $\log 432 = \log (2^4 \times 3^3) = 4 \log 2 + 3 \log 3 = 2.6353$.

Then, by § 92, the *mantissa* of the result is .6353.

Whence, by § 83, $\log .00432 = 7.6353 - 10$.

EXERCISE 15

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find:

1. $\log 2.7$.
2. $\log 14.7$.
3. $\log .56$.
4. $\log .0162$.
5. $\log 22.5$.
6. $\log .00000686$.
7. $\log .00125$.
8. $\log 5670$.
9. $\log .0000588$.
10. $\log .000864$.
11. $\log 337.5$.
12. $\log 3.888$.
13. $\log (4.5)^8$.
14. $\log \sqrt[6]{8.4}$.
15. $\log (24.3)^{\frac{3}{2}}$.

USE OF THE TABLE

94. The table (pages 50 and 51) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

95. *To find the logarithm of a number of three figures.*

Look in the column headed "No." for the first two significant figures of the given number.

Then the required mantissa will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic in accordance with the rules of §§ 82 and 83.

For example, $\log 168 = 2.2253$;
$\log .344 = 9.5366 - 10$; etc.

For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find $\log 83$ and $\log 9$.

By § 92, $\log 83$ has the same mantissa as $\log 830$, and $\log 9$ the same mantissa as $\log 900$.

Hence, $\log 83 = 1.9191$, and $\log 9 = 0.9542$.

96. *To find the logarithm of a number of more than three figures.*

1. Required the logarithm of 327.6.

We find from the table, $\log 327 = 2.5145,$
$\log 328 = 2.5159.$

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Then an increase of .6 of a unit in the number will increase the logarithm by $.6 \times .0014$, or .0008 to the nearest fourth decimal place.

Whence, $\log 327.6 = 2.5145 + .0008 = 2.5153.$

In finding the logarithm of a number, the difference between the next less and next greater mantissæ is called the *tabular difference;* thus, in Ex. 1, the tabular difference is .0014.

The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

In finding the correction to the nearest units' figure, the decimal portion should be omitted, provided that if it is .5, or greater than .5, the units' figure is increased by 1; thus, 13.26 would be taken as 13, 30.5 as 31, and 22.803 as 23.

2. Find the logarithm of .021508.

Mantissa 215 = .3324	Tab. diff. = 21
2	.08
.3326	Correction = 1.68 = 2, nearly.

The result is $8.3326 - 10$.

EXERCISE 16

Find the logarithms of the following:

1. 64.	5. 1079.	9. .00005023.	13. 7.3165.
2. 3.7.	6. .6757.	10. .0002625.	14. .019608.
3. 982.	7. .09496.	11. 31.393.	15. 810.39.
4. .798.	8. 4.288.	12. 48387.	16. .0025446.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

97. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two places to the left of the decimal point (§ 82).

Hence, the number corresponding to 1.6571 is 45.4.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945.

The numbers corresponding to the logarithms 2.3927 and 2.3945 are 247 and 248, respectively.

That is, an increase of .0018 in the mantissa produces an increase of one unit in the number corresponding.

Then, an increase of .0007 in the mantissa will increase the number by $\frac{7}{18}$ of a unit, or .4, nearly.

Hence, the number corresponding is 247 + .4, or 247.4.

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

The rules for pointing off are the reverse of those of §§ 82 and 83:

I. *If* -10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of places to the left of the decimal point.*

II. *If* -10 *is written after the mantissa, subtract the positive part of the characteristic from* 9, *giving the number of ciphers to be placed between the decimal point and first significant figure.*

3. Find the number whose logarithm is 8.5265 − 10.

5265

Next less mant. = 5263; figures corresponding, 336.

Tab. diff. 13)2.00(.15 = .2, nearly.

1 3

70

By the above rule, there will be one cipher to be placed between the decimal point and first significant figure; the result is .03362.

The correction can usually be depended upon to only one decimal place; the division should be carried to two places to determine the last figure accurately.

EXERCISE 17

Find the numbers corresponding to the following logarithms:

1. 0.8189.
2. 7.6064 – 10.
3. 1.8767.
4. 2.6760.
5. 3.9826.
6. 8.7954 – 10.
7. 6.5993 – 10.
8. 9.9437 – 10.
9. 0.7781.
10. 5.4571 – 10.
11. 1.3019.
12. 4.2527 – 10.
13. 2.0159.
14. 3.7264 – 10.
15. 4.4929.

APPLICATIONS

98. The *approximate* value of a number in which the operations indicated involve only multiplication, division, involution, or evolution may be conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

1. Find the value of $.0631 \times 7.208 \times .51272$.

By § 87, $\log(.0631 \times 7.208 \times .51272)$

$$= \log .0631 + \log 7.208 + \log .51272.$$

$$\begin{aligned} \log .0631 &= 8.8000 - 10 \\ \log 7.208 &= 0.8578 \\ \log .51272 &= 9.7099 - 10 \end{aligned}$$

Adding, log of result $= 19.3677 - 20 = 9.3677 - 10$. (See Note 1.)

Number corresponding to $9.3677 - 10 = .2332$.

Note 1: If the sum is a negative logarithm, it should be written in such a form that the negative portion of the characteristic may be – 10.

Thus, $19.3677 - 20$ is written $9.3677 - 10$.

(In computations with four-place logarithms, the result cannot usually be depended upon to more than *four* significant figures.)

2. Find the value of $\frac{336.8}{7984}$.

By § 88, $\log \frac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\begin{aligned} \log 336.8 &= 12.5273 - 10 \\ \log 7984 &= 3.9022 \end{aligned}$$

Subtracting, $\quad$ log of result $= 8.6251 - 10$ (See Note 2.)

Number corresponding $= .04218.$

Note 2: To subtract a greater logarithm from a less, or a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By § 89, $\log (.07396)^5 = 5 \times \log .07396.$

$$\begin{aligned} \log .07396 &= 8.8690 - 10 \\ & 5 \\ & 44.3450 - 50 \\ &= 4.3450 - 10 = \log .000002213. \end{aligned}$$

4. Find the value of $\sqrt[3]{.035063}$.

By § 90, $\log \sqrt[3]{.035063} = \frac{1}{3} \log .035063.$

$$\begin{aligned} \log .035063 &= 8.5449 - 10 \\ 3)&\underline{28.5449 - 30} \\ & 9.5150 - 10 = \log .3224. \end{aligned}$$

(See Note 3.)

Note 3: To divide a negative logarithm, write it in such a form that the negative portion of the characteristic may be exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, we write the logarithm in the form $28.5449 - 30$; dividing this by 3, the quotient is $9.5150 - 10$.

EXERCISE 18

A *negative* number has no common logarithm (§ 78); if such numbers occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3 of the following set, to find the value of $(-95.86) \times 3.3918$ we find the value of 95.86×3.3918, and put a $-$ sign before the result.

Find by logarithms the values of the following:

1. 4.253×7.104.

2. $6823.2 \times .1634$.

3. $(-95.86) \times 3.3918$.

4. $54.029 \times (-.0081487)$.

5. $.040764 \times .12896$.

6. $(-285.46) \times (-.00070682)$.

7. $\frac{5978}{9.762}$.

8. $\frac{21.658}{45057}$.

9. $\frac{.06405}{.002037}$.

10. $\frac{-38.19}{.10792}$.

11. $\frac{670.43}{-5382.3}$.

12. $\frac{.000007913}{.00082375}$.

13. $(88.08)^3$.

14. $(.09437)^4$.

15. $(3.625)^7$.

Arithmetical Complement

99. The **Arithmetical Complement** of the logarithm of a number, or, briefly, the **Cologarithm** of the number, is the logarithm of the reciprocal of that number.

Thus, $\text{colog } 409 = \log \frac{1}{409} = \log 1 - \log 409.$

$$\begin{aligned} \log 1 &= 10. \quad -10 \quad \text{(See Ex. 2, § 98.)} \\ \log 409 &= \underline{2.6117} \\ \therefore \text{colog } 409 &= 7.3883 - 10. \end{aligned}$$

Again, $\text{colog } .067 = \log \frac{1}{.067} = \log 1 - \log .067$

$$\begin{aligned} \log 1 &= 10. \quad -10 \\ \log .067 &= \underline{8.8261 - 10} \\ \therefore \text{colog } .067 &= 1.1739. \end{aligned}$$

It follows from the above that *the cologarithm of a number may be found by subtracting its logarithm from* 10 — 10.

The cologarithm may be found by subtracting the last *significant* figure of the logarithm from 10 and each of the others from 9, — 10 being written after the result in the case of a positive logarithm.

Ex. Find the value of $\frac{.51384}{8.708 \times .0946}$.

$$\log \frac{.51384}{8.708 \times .0946} = \log \left(.51384 \times \frac{1}{8.708} \times \frac{1}{.0946}\right)$$
$$= \log .51384 + \log \frac{1}{8.708} + \log \frac{1}{.0946}$$
$$= \log .51384 + \text{colog } 8.708 + \text{colog } .0946.$$

$$\log .51384 = 9.7109 - 10$$
$$\text{colog } 8.708 = 9.0601 - 10$$
$$\text{colog } .0946 = 1.0241$$
$$9.7951 - 10 = \log .6239.$$

It is evident from the above example that, to find the logarithm of a fraction whose terms are the products of factors, we **add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.**

The value of the above fraction may be found without using cologarithms, by the following formula:

$$\log \frac{.51384}{8.709 \times .0946} = \log .51384 - \log(8.709 \times .0946)$$
$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

MISCELLANEOUS EXAMPLES

100. 1. Find the value of $\frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \text{colog } 3^{\frac{5}{6}} \quad (\S\ 99)$$
$$= \log 2 + \tfrac{1}{3} \log 5 + \tfrac{5}{6} \text{colog } 3.$$

$$\log 2 = .3010$$
$$\log 5 = .6990; \quad \div 3 = .2330$$
$$\text{colog } 3 = 9.5229 - 10; \quad \times \tfrac{5}{6} = 9.6024 - 10$$
$$.1364 = \log 1.369.$$

2. Find the value of $\sqrt[3]{\frac{-.03296}{7.962}}$.

$$\log\sqrt[3]{\frac{.03296}{7.962}} = \frac{1}{3}\log\frac{.03296}{7.962} = \frac{1}{3}(\log .03296 - \log 7.962).$$

$$\begin{aligned} \log .03296 &= 8.5180 - 10 \\ \log\ 7.962 &= 0.9010 \\ 3)&\underline{27.6170 - 30} \\ &9.2057 - 10 = \log .1606. \end{aligned}$$

The result is $-.1606$.

EXERCISE 19

Find by logarithms the values of the following:

1. $\frac{2078.5 \times .05834}{.3583 \times 346}$.

2. $\frac{(-6.08) \times .1304}{4.046 \times .0031095}$.

3. $\frac{(-.076917) \times 26.3}{.5478 \times (-3120.7)}$.

4. $\frac{.8102 \times (-6.225)}{(-0721) \times (-17.976)}$.

5. $6^{\frac{6}{5}} \times 5^{\frac{5}{6}}$.

6. $\frac{7^{\frac{4}{9}}}{9^{\frac{3}{8}}}$.

7. $\sqrt[11]{\frac{68}{35}}$.

8. $\frac{\sqrt[7]{8}}{(.1)^{\frac{2}{3}}}$.

9. $\frac{(100)^{\frac{2}{7}}}{\sqrt[3]{-.004}}$.

10. $(-\frac{5510}{7048})^{\frac{4}{3}}$.

11. $\sqrt{\frac{38.7}{501.9}}$.

12. $\frac{\sqrt[5]{-.01}}{4^{\frac{9}{4}}}$.

13. $\frac{-(.03)^{\frac{5}{2}}}{\sqrt[9]{-1000}}$.

14. $\sqrt[4]{\frac{7}{9}} \div \sqrt[8]{\frac{3}{4}}$.

15. $\sqrt{6} \times \sqrt[6]{10} \times \sqrt[10]{2}$.

16. $\left(-\frac{24.18}{8.7 \times .0603}\right)^{\frac{3}{5}}$.

17. $\frac{\sqrt[5]{.008546}}{\sqrt[6]{.0003867}}$.

18. $\frac{(-.14582)^{\frac{5}{3}}}{-(.72346)^{\frac{7}{4}}}$.

IV. FACTORS

101. An **irrational number** is a numerical expression involving surds; as $\sqrt[3]{3}$, or $2 + \sqrt{5}$ (§ 70).

102. A rational and integral expression is resolved into its prime factors when further factoring would produce irrational factors.

103. In the First Course we considered the following eight types of factorable numbers:

TYPE FORMS

I. $a^2 - b^2 = (a+b)(a-b)$.

II. $a^2 + 2ab + b^2 = (a+b)(a+b)$,
$a^2 - 2ab + b^2 = (a-b)(a-b)$.

III. $x^2 + ax + b$.

IV. $ax^2 + bx + c$.

V. $x^4 + ax^2y^2 + y^4$.

VI. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$,
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

VII. $a^n - b^n$,
$a^n + b^n$.

VIII. $ax + ay + az = a(x + y + z)$.

Of these types, IV is more readily factored by means of VIII as follows:

Ex. Factor $6x^2 - 7x - 20$.

Multiply -20 by 6 (the coefficient of x^2). Factor -120 so that the sum of the factors is -7 (the coefficient of x). These factors are -15, 8. Then write

$$6x^2 - 7x - 20 = 6x^2 - 15x + 8x - 20.$$

Group by Type VIII, $\quad = 3x(2x-5) + 4(2x-5)$,

whence, $\quad 6x^2 - 7x - 20 = (2x-5)(3x+4)$.

Type VI may be placed under Type VII.

EXERCISE 20

Factor:

1. $3x^2 - x - 10$.
2. $4a^2 + 12a + 9$.
3. $x^3 - y^3$.
4. $a^3 + a^2 - 2a - 2$.
5. $x^4 + 4$.
6. $x^3 + 8$.
7. $a^2 + 9b^2 - 4c^2 + 6ab$.
8. $x^2 + 2xy + y^2 + 8(x+y) + 16$.

9. $x^{10} - 2x^5 + 1$.

10. $6a^2 - 17a + 12$.

11. $9x^4 - 13x^2 + 4$.

12. $x^6 + 7x^3 - 8$.

13. $(a-b)^2 - 2(a-b) - 35$.

14. $m^2 + (a-b)m - ab$.

15. $m^{\frac{3}{2}} - 1$.

16. $(2a - 3b)^2 - (a-b)^2$.

17. $p^{-2} - 7p^{-1} + 12$.

18. $a^3 + b + a + b^3$.

19. $x^3 + 3x^2 + 3x + 1$.

20. $9m^2 - 36mn$.

21. $x^9 - x^3 + x^6 - 1$

22. $6a^2b - 4a^2 + 15ab - 10a$.

23. $a^{-\frac{5}{2}} - 32$.

24. $9x^{-4} + 12x^{-2} + 4$.

25. $9a^2 - 30ab + 25b^2 - 4c^2$.

26. $9a^2 - 25c^2 + b^2 + 6ab$.

27. $36x^4 - 61x^2 + 25$.

28. $(3a-b)^2 - 6x(3a-b) + 27a - 9b$.

29. $2a^{-6} + 250$.

30. $(7x+2) + 3\sqrt{7x+2} + 2$.

31. $m(2x-3) - 4m^2x^2 + 9m^2$.

32. $x^6 + y^6$.

33. $16x^{\frac{4}{3}} + 14x^{\frac{2}{3}} - 15$.

34. $25(m+3)^2 + 10(m+3) + 1$.

35. $8(2a-5b)^{-1} - 12(2a-5b)^{-\frac{1}{2}} + 4$.

36. $143k^2 - 103k + 14$.

37. $\sqrt{x^2 + 4x - 6} + 2x^2 - 1 + 4(2x-3)$.

38. $g^4 + g^2t^2 + t^4$.

39. $x^3 + a^2x - a^3 - ax^2$.

40. $c^{-\frac{1}{3}}d - 18 - 9d + 2c^{-\frac{1}{3}}$.

41. $r^4 - 20r^2 + 99$.

42. $x^{12} + y^{12}$.

43. $x^4 - 13x^2 + 4$.

44. $9e^3f - 16ef^3$.

45. $304v^2 + 25v - 6$.

46. $(x+1)^{\frac{2}{3}} + 2(x+1)^{\frac{1}{3}} + 1$.

47. $5(x^2+y^2)^3 + 6(x^2+y^2)^2 + (x^2+y^2)$.

48. $a^6 - 64$.

49. $6x^{-4} - 41x^{-2}y^{\frac{1}{2}} - 7y$.

50. $25a^4 + a^2 + 1$.

51. $4 + a^3 - a^2 - 4a$.

52. $m^4 - 1 + m - m^3$.

53. $3(x^3+1) + 5(x^2-1) + (x+1)^2$.

54. $6x^{-2} + 13x^{-1} + 6$.

55. $a^{\frac{5}{3}} + b^{\frac{5}{3}}$.

56. $x^{10} - y^5z^5$.

57. $p^2 - q$.

58. $4a^{2x+4} - 4a^{x+2} + 1$.

59. $a^2x - 9x + 2a^2 - 18$.

60. $4p^2q^2 + 20pq^2 - 16p^2q - 80pq.$

61. $(x^2 - 2x + 1) - (x + 1)^2.$

62. $52m - 10m^2 - 10.$

63. $x^{3m} - y^{3n}.$

64. $a^8 - 256.$

65. $a^5 - 27a^2 + 243 - 9a^3.$

66. $2^{2m} + 4x \cdot 2^m - 21x^2.$

67. $a + 2\sqrt{ab} + b.$

68. $(2a - 3b)^2 - (3a - 2b)^2.$

69. $a^2 + 2ab + c^2 - 2bc - 2ac + b^2.$

70. $27m^3 - 54m^2 + 36m - 8.$

71. $x^5 + x^4 + x^3 + x^2 + x + 1.$

72. $9a^2 - 6a - 4b^2 - 4b.$

73. $1 + 2ab - a^4 - a^2b^2 - b^4.$

74. $am^2 - m^3 + 2amn + an^2 + 2m^2n - mn^2.$

75. $y^2 + y^2z - 2y - z + 1.$

FACTOR THEOREM

104. The Remainder Theorem.

Let it be required to divide $px^2 + qx + r$ by $x - a$.

$$\begin{array}{l|l} px^2 + qx + r & x - a \\ \cline{2-2} \underline{px^2 - apx} & px + (ap + q) \\ (ap + q)x & \\ \underline{(ap + q)x - pa^2 - qa} & \\ \qquad pa^2 + qa + r, \text{ Remainder.} & \end{array}$$

We observe that the final remainder,

$$pa^2 + qa + r,$$

is the same as the dividend with a substituted in place of x; this exemplifies the following law:

If any polynomial, involving x, be divided by $x - a$, the remainder of the division equals the result obtained by substituting a for x in the given polynomial.

This is called *The Remainder Theorem.*

To prove the theorem, let

$$px^n + qx^{n-1} + \cdots + rx + s$$

be any polynomial involving x.

Let the division of the polynomial by $x - a$ be carried on until a remainder is obtained which does not contain x.

Let Q denote the quotient, and R the remainder.

Since the dividend equals the product of the quotient and divisor, plus the remainder, we have

$$Q(x-a)+R=px^n+qx^{n-1}+\cdots+rx+s.$$

Putting x equal to a, into the above equation, we have,

$$R=pa^n+qa^{n-1}+\cdots+ra+s.$$

105. The Factor Theorem.

If any polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x-a$ as a factor.

For, by § 104, if the polynomial is divided by $x-a$, the remainder is zero.

106. Examples.

1. Find whether $x-2$ is a factor of x^3-5x^2+8.

Substituting 2 for x, the expression x^3-5x^2+8 becomes

$$2^3-5\cdot 2^2+8, \text{ or } -4.$$

Then, by § 104, if x^3-5x^2+8 be divided by $x-2$, the remainder is -4; and $x-2$ is not a factor.

2. Find whether $m+n$ is a factor of

$$m^4-4m^3n+3m^2n^2+5mn^3-2n^4. \qquad (1)$$

Putting $m=-n$, the expression becomes

$$n^4+4n^4+2n^4-5n^4-2n^4, \text{ or } 0.$$

Then, by § 104, if the expression (1) be divided by $m+n$, the remainder is 0; and $m+n$ is a factor.

3. Prove that a is a factor of

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a).$$

Putting $a=0$, the expression becomes

$$(b+c)bc-b(b+c)c, \text{ or } 0.$$

Then, by § 104, $a-0$, or a, is a factor of the expression.

4. Factor $x^3 - 3x^2 - 14x - 8$.

The positive and negative integral factors of 8 are 1, 2, 4, 8, -1, -2, -4, and -8.

It is best to try the numbers in their order of absolute magnitude.

If $x = 1$, the expression becomes $1 - 3 - 14 - 8$.

If $x = -1$, the expression becomes $-1 - 3 + 14 - 8$.

If $x = 2$, the expression becomes $8 - 12 - 28 - 8$.

If $x = -2$, the expression becomes $-8 - 12 + 28 - 8$, or 0.

This shows that $x + 2$ is a factor.

Dividing the expression by $x + 2$, the quotient is $x^2 - 5x - 4$.

Then, $$x^3 - 3x^2 - 14x - 8 = (x + 2)(x^2 - 5x - 4).$$

EXERCISE 21

Factor the following:

1. $a^3 + 8$.
2. $m^5 + n^5$.
3. $x^6 - 729$.
4. $x^3 + 5x^2 - 8x + 2$.
5. $m^3 - 11m - 10$.
6. $a^4 - a^3 + 3a - 14$.
7. $c^3 - 2c^2 - 9$.
8. $x^4 - 625$.
9. $a^n - b^n$.
10. $2x^3 + 5x^2 - x - 6$.
11. $x^4 - x^3 + 2x^2 - 4$.
12. $5a^3 - 18a - 4$.
13. $x^3 + x^2 + 7x + 18$.
14. $m^3 - 5m^2 - 36$.
15. $k^4 - 5k^2 + 3k - 2$.

Find without actual division:

16. Whether $p - 1$ is a factor of $p^3 + 3p^2 - 4$.
17. Whether $x + 2$ is a factor of $x^4 + 3x^3 - 4x$.
18. Whether $x + 1$ is a factor of $2x^3 + 5x^2 - 3x + 4$.
19. Whether $m - 3$ is a factor of $m^3 - 4m - 15$.
20. Whether $a - 5$ is a factor of $a^3 - 3a^2 - 5a - 25$.
21. Whether $c - 2$ is a factor of $3c^3 - 9c^2 + 5c + 2$.
22. Whether a is a factor of $a(b - c) + b(c - a) + c(a - b)$.
23. Whether c is a factor of $a(b - c) + b(c - a) + c(a - b)$.
24. Whether $x + y$ is a factor of $x(2x + 3y) - y(3x + 2y)$.
25. Whether b is a factor of $a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2$.

HORNER'S SYNTHETIC DIVISION

107. The method of synthetic division, or as it is sometimes known, the method of detached coefficients, greatly abridges the work of division, especially where binomial divisors are concerned.

108. Divide $x^3 - 11\,x^2 + 36\,x - 36$ by $x - 3$.

Writing dividend and divisor with coefficients only,

$$\begin{array}{llll|l}
1 & -\,11 & +\,36 & -\,36 & 1 - 3 \\ \cline{5-5}
\underline{1} & \underline{-\;\;3} & & & 1 - 8 + 12 \quad \textit{Quotient.} \\
 & -\;\;8 & & & \\
 & \underline{-\;\;8} & \underline{+\,24} & & \\
 & & +\,12 & -\,36 & \\
 & & \underline{+\,12} & \underline{-\,36} &
\end{array}$$

Since the first term of each partial product is merely a repetition of the term immediately above, it may be omitted.

We may also change the sign of the second term of the divisor if the partial product is added instead of subtracted.

We then have

$$\begin{array}{llll|l}
1 & -\,11 & +\,36 & -\,36 & 1 + 3 \\ \cline{5-5}
\underline{1} & \underline{+\,3} & & & 1 - 8 + 12 \\
 & -\,8 & & & \\
 & & \underline{-\,24} & & \\
 & & +\,12 & & \\
 & & & \underline{+\,36} &
\end{array}$$

Raise the numbers -24, 36 now in the oblique column and the work stands :

$$\begin{array}{llll|l}
1 & -\,11 & +\,36 & -\,36 & +\,3 \\ \cline{5-5}
 & \underline{+\;\;3} & \underline{-\,24} & \underline{+\,36} & \\
 & - & +\,12 & &
\end{array}$$

The quotient is $x^2 - 8\,x + 12$.

If the last remainder is zero, x minus the divisor is a factor of the expression.

EXERCISE 22

Divide the following by synthetic division:

1. $2x^3 - 7x^2 + x + 10$ by $x - 2$.

2. $3a^4 - a^3 - 5a^2 + 6a + 7$ by $a + 1$.

3. $a^4 - 11a^3 + 29a^2 - 9a + 14$ by $a - 7$.

4. $4m^3 - 17m^2n + 13mn^2 + 6n^3$ by $m - 3n$.

5. $3x^5 + 11x^4 - 43x^3 - 4x^2 + 11x - 6$ by $x + 6$.

6. $8v^4 - 35v^3 + 7v^2 + 22v - 8$ by $v - 4$.

109. Divide $x^3 - 11x^2 + 36x - 36$ by $x - 5$, and by $x - 7$.

$$\begin{array}{l} 1 - 11 + 36 - 36 \;\underline{|5} \\ \quad + \;5 - 30 + 30 \\ \hline \quad - \;6 + \;6 - \;\;6 \quad \text{Remainder} \\ \text{(Quotient)} \end{array} \qquad \begin{array}{l} 1 - 11 + 36 - 36 \;\underline{|7} \\ \quad + \;7 - 28 + 56 \\ \hline \quad - \;4 + \;8 + 20 \quad \text{Remainder} \\ \text{(Quotient)} \end{array}$$

A factor lies between $x - 5$ and $x - 7$. It is found to be $x - 6$.

Then if in dividing by a binomial a remainder occurs, and if the remainders arising from successive division by two binomials are of opposite sign, a factor $x - a$ lies between these two binomials.

EXERCISE 23

1. Locate the root between 2 and 4 of $x^3 - 17x + 24 = 0$.

Locate roots of the following:

2. $a^3 + 10a^2 + 17a - 28 = 0$.

3. $a^4 + 3a^3 - 10a^2 + 3a + 15 = 0$.

4. $x^5 - 8x^4 - 7x^3 + 56x^2 - 5x + 40 = 0$.

5. $3x^3 - 26x^2 + 60x - 72 = 0$.

6. $m^4 - 2m^3 - 19m^2 + 12m + 40 = 0$.

SOLUTIONS

110. If the product of $abc \cdots$ to n factors $= 0$, at least one of the factors must be zero.

Ex. 1. Let $(x-2)(x-3)(x+4)=0$.

Then $x-2$, $x-3$, or $x+4$ must equal zero.

The equation is satisfied by the root obtained by putting any one of the factors equal to 0. Hence, $x = 2, 3$, or -4 are the solutions of the equation.

Ex. 2. Solve $$5^{2x} - 5^x - 12 = 0. \quad (1)$$

$$(5^x - 4)(5^x + 3) = 0. \quad (2)$$

Whence, $$5^x - 4 = 0, \quad 5^x = 4, \quad (3)$$

and $$5^x + 3 = 0, \quad 5^x = -3. \quad (4)$$

To solve (3) and (4), take the logarithms of each member of the equations:

From (3) $$x \log 5 = \log 4 \quad (\S\ 89), \quad (5)$$

and $$x = \frac{\log 4}{\log 5} = \frac{.6020}{.6990} = \frac{602}{699}. \quad (6)$$

From (4) $$x \log 5 = -\log 3.$$

$$x = -\tfrac{4771}{6990}.$$

Ex. 3. Solve the equation $.2^x = 3$.

Taking the logarithms of both members, $x \log .2 = \log 3$.

Then $$x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010 - 10} = \frac{.4771}{-.699} = -.6285+.$$

An equation of the form $a^x = b$ may be solved by inspection if b can be expressed as an exact power of a.

Ex. 4. Solve the equation $16^x = 128$.

We may write the equation $(2^4)^x = 2^7$, or $2^{4x} = 2^7$.

Then, by inspection, $4x = 7$; and $x = \frac{7}{4}$.

(If the equation were $16^x = \dfrac{1}{128}$, we could write it $(2^4)^x = \dfrac{1}{2^7} = 2^{-7}$; then $4x$ would equal -7, and $x = -\frac{7}{4}$.)

EXERCISE 24

Solve the following equations:

1. $13^x = 8$.
2. $.06^x = .9$.
3. $9.347^x = .0625$.
4. $.005038^x = 816.3$.
5. $3^{4x-1} = 4^{2x+3}$.
6. $7^{3x+2} = .8^x$.
7. $.2^{x+5} = .5^{x-4}$.
8. $16^x = 32$.
9. $32^x = \frac{1}{64}$.
10. $(\frac{1}{16})^x = 8$.
11. $(\frac{1}{9})^x = \frac{1}{27}$.
12. $.04^{2x} - 5(.04)^x - 24 = 0$.
13. $2^{3x} + 7.2^{2x} - 9.2^x - 63 = 0$.
14. $3^{3y} - 5.3^{2y} - 8.3^y + 12 = 0$.
15. $11^{4x} - 5 \cdot 11^{2x} + 4 = 0$.
16. $2^{3x+3} - 6 \cdot 2^{2x+2} + 11 \cdot 2^{x+1} - 6 = 0$.
17. $.5^{4x} - 2(.5)^{3x} - 16(.5)^{2x} + 2(.5)^x + 15 = 0$.
18. $2^{3x} - 10 \cdot 2^{2x} - 71 \cdot 2^x - 60 = 0$.
19. $x^3 - x^2 - 9x + 9 = 0$.
20. $x^2 + (5c + 2d)x + 10cd = 0$.

COMMON FACTORS AND MULTIPLES

111. A **Common Factor** of two or more expressions is a factor of each of them.

112. The **Highest Common Factor** (H. C. F.) of two or more expressions is their common factor of *highest degree* (§ 23).

113. A **Common Multiple** of two or more expressions is an expression which is exactly divisible by each of them.

114. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is their common multiple of *lowest degree*.

Ex. 1. Find the H. C. F. of $a^2 + 2a - 3$ and $1 - a^3$.

$$a^2 + 2a - 3 = (a - 1)(a + 3).$$
$$1 - a^3 = (1 - a)(1 + a + a^2).$$

The factors of the first expression can be put in the form

$$-(1 - a)(3 + a).$$

Hence, the H. C. F. is $1 - a$.

Ex. 2. Required the L. C. M. of

$$x^2 - 5x + 6,\ x^2 - 4x + 4,\ \text{and}\ x^3 - 9x.$$

$$x^2 - 5x + 6 = (x-3)(x-2).$$
$$x^2 - 4x + 4 = (x-2)^2.$$
$$x^3 - 9x = x(x+3)(x-3).$$

It is evident by inspection that the L. C. M. of these expressions is

$$x(x-2)^2(x+3)(x-3).$$

115. When the polynomials cannot be readily factored by inspection, the H. C. F. and L. C. M. may be found by the following method.

The rule in Arithmetic for the H. C. F. of two numbers is:

Divide the greater number by the less.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. required.

Thus, let it be required to find the H. C. F. of 169 and 546.

```
169)546(3
    507
     39)169(4
        156
         13)39(3
            39
```

Then 13 is the H. C. F. required.

116. We will now prove that a rule similar to that of § 115 holds for the H. C. F. of two algebraic expressions.

Let A and B be two polynomials, arranged according to the descending powers of some common letter.

Let the exponent of this letter in the first term of A be equal to, or greater than, its exponent in the first term of B.

Suppose that B is contained in A p times, with a remainder C; that C is contained in B q times, with a remainder D; and that D is contained in C r times, with no remainder.

To prove that D is the H. C. F. of A and B.

The operation of division is shown as follows:

$$\begin{array}{l} B)A(p \\ \underline{pB} \\ \quad C)B(q \\ \quad \underline{qC} \\ \qquad D)C(r \\ \qquad \underline{rD} \\ \qquad\quad 0 \end{array}$$

We will first prove that D is a common factor of A and B.

Since the minuend is equal to the subtrahend plus the remainder (F. C., § 40),

$$A = pB + C, \quad (1)$$

$$B = qC + D, \quad (2)$$

and

$$C = rD.$$

Substituting the value of C in (2), we obtain

$$B = qrD + D = D(qr + 1). \quad (3)$$

Substituting the values of B and C in (1), we have,

$$A = pD(qr + 1) + rD = D(pqr + p + r). \quad (4)$$

From (3) and (4), D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let F be any common factor of A and B; and let

$$A = mF, \text{ and } B = nF.$$

From the operation of division, we have

$$C = A - pB, \quad (5)$$

and

$$D = B - qC. \quad (6)$$

Substituting the values of A and B in (5), we have

$$C = mF - pnF.$$

Substituting the values of B and C in (6) we have

$$D = nF - q(mF - pnF) = F(n - qm + pqn).$$

Whence, F is a factor of D.

Then, since every common factor of A and B is a factor of D, and since D itself is a common factor of A and B, it follows that D is the *highest* common factor of A and B.

We then have the following rule for the H. C. F. of two polynomials, A and B, arranged according to the descending powers of some common letter, the exponent of that letter in the first term of A being equal to, or greater than, its exponent in the first term of B:

Divide A by B.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. required.

It is important to keep the work throughout in descending powers of some common letter; and each division should be continued until the exponent of this letter in the first term of the remainder is less than its exponent in the first term of the divisor.

Note 1: If the terms of one of the expressions have a common factor which is not a common factor of the terms of the other, it may be removed; for it can evidently form no part of the highest common factor.

In like manner, we may divide any remainder by a factor which is not a factor of the preceding divisor.

117. 1. Find the H. C. F. of

$$6x^2-25x+14 \text{ and } 6x^3-7x^2-25x+18.$$

$$
\begin{array}{r|l}
6x^2-25x+14)\,6x^3-7x^2-25x+18\,(x+3 & \\
6x^3-25x^2+14x & \\
\hline
18x^2-39x+18 & \\
18x^2-75x+42 & \\
\hline
36x-24 &
\end{array}
$$

In accordance with Note 1, we divide this remainder by 12, giving $3x-2$.

$$
\begin{array}{r}
3x-2)\,6x^2-25x+14\,(2x-7 \\
6x^2-4x \\
\hline
-21x+14 \\
-21x+14 \\
\hline
\end{array}
$$

Then, $3x-2$ is the H. C. F. required.

Note 2: If the first term of the dividend, or of any remainder, is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any term which is not a factor of the divisor.

2. Find the H. C. F. of

$$3\,a^3 + a^2b - 2\,ab^2 \text{ and } 4\,a^3b + 2\,a^2b^2 - ab^3 + b^4.$$

We remove the factor a from the first expression and the factor b from the second (Note 1), and find the H. C. F. of

$$3\,a^2 + ab - 2\,b^2 \text{ and } 4\,a^3 + 2\,a^2b - ab^2 + b^3.$$

Since $4\,a^3$ is not divisible by $3\,a^2$, we multiply the second expression by 3 (Note 2).

$$\begin{array}{r|l}
 & 4\,a^3 + 2\,a^2b - ab^2 + b^3 \\
 & \hphantom{4\,a^3 + 2\,a^2b - ab^2 + 3}3 \\
\hline
3\,a^2 + ab - 2\,b^2) & 12\,a^3 + 6\,a^2b - 3\,ab^2 + 3\,b^3(4\,a \\
 & 12\,a^3 + 4\,a^2b - 8\,ab^2 \\
\hline
 & \phantom{12\,a^3 +{}} 2\,a^2b + 5\,ab^2 + 3\,b^3
\end{array}$$

Since $2\,a^2b$ is not divisible by $3\,a^2$, we multiply this remainder by 3 (Note 2).

$$\begin{array}{r|l}
 & 2\,a^2b + 5\,ab^2 + 3\,b^3 \\
 & \hphantom{2\,a^2b + 15\,ab^2 + 9}3 \\
\hline
3\,a^2 + ab - 2\,b^2) & 6\,a^2b + 15\,ab^2 + 9\,b^3(2\,b \\
 & 6\,a^2b + 2\,ab^2 - 4\,b^3 \\
\hline
 & \phantom{6\,a^2b +{}} 13\,ab^2 + 13\,b^3
\end{array}$$

We divide this remainder by $13\,b^2$ (Note 1), giving $a + b$.

$$\begin{array}{r|l}
a + b) & 3\,a^2 + ab - 2\,b^2(3\,a - 2\,b \\
 & 3\,a^2 + 3\,ab \\
\hline
 & - 2\,ab \\
 & - 2\,ab - 2\,b^2 \\
\hline
\end{array}$$

Then, $a + b$ is the H. C. F. required.

Note 3: If the first term of any remainder is negative, the sign of each term of the remainder may be changed.

Note 4: If the given expressions have a common factor which can be seen by inspection, remove it, and find the H. C. F. of the resulting expressions; the result, multiplied by the common factor, will be the H. C. F. of the given expressions.

3. Find the H. C. F. of

$$2x^4 + 3x^3 - 6x^2 + 2x \text{ and } 6x^4 + 5x^3 - 2x^2 - x.$$

Removing the common factor x (Note 4), we find the H. C. F. of

$$2x^3 + 3x^2 - 6x + 2 \text{ and } 6x^3 + 5x^2 - 2x - 1.$$

$$\begin{array}{r}
2x^3 + 3x^2 - 6x + 2)\,6x^3 + 5x^2 - 2x - 1(3 \\
\underline{6x^3 + 9x^2 - 18x + 6} \\
-4x^2 + 16x - 7
\end{array}$$

The first term of this remainder being negative, we change the sign of each of its terms (Note 3).

$$\begin{array}{r}
2x^3 + 3x^2 - 6x + 2 \\
\underline{2} \\
4x^2 - 16x + 7)\,4x^3 + 6x^2 - 12x + 4(x \\
\underline{4x^3 - 16x^2 + 7x} \\
22x^2 - 19x + 4 \\
\underline{2} \\
44x^2 - 38x + 8(11 \\
\underline{44x^2 - 176x + 77} \\
69)\,138x - 69 \\
2x - 1 \\
2x - 1)\,4x^2 - 16x + 7(2x - 7 \\
\underline{4x^2 - 2x} \\
-14x \\
\underline{-14x + 7}
\end{array}$$

The last divisor is $2x - 1$; multiplying this by x, the H. C. F. of the given expressions is $x(2x - 1)$.

(In the above solution, we multiply $2x^3 + 3x^2 - 6x + 2$ by 2 in order to make its first term divisible by $4x^2$; and we multiply the remainder $22x^2 - 19x + 4$ by 2 to make its first term divisible by $4x^2$.)

118. We will now show how to find the L. C. M. of two expressions which cannot be readily factored by inspection.

Let A and B be any two expressions.

Let F be their H. C. F., and M their L. C. M.

Suppose that $A = aF$, and $B = bF$.

Then, $$A \times B = abF^2. \qquad (1)$$

Since F is the H. C. F. of A and B, a and b have no common factors; whence the L. C. M. of aF and bF is abF.

That is, $$M = abF.$$

Multiplying each of these equals by F, we have

$$F \times M = abF^2. \qquad (2)$$

From (1) and (2), $A \times B = F \times M.$

That is, *the product of two expressions is equal to the product of their* H. C. F. *and* L. C. M.

Therefore, to find the L. C. M. of two expressions,

Divide their product by their highest common factor; or,

Divide one of the expressions by their highest common factor, and multiply the quotient by the other expression.

Ex. Find the L. C. M. of

$$6x^2 - 17x^2 + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{l}
6x^2 - 17x + 12)12x^2 - 4x - 21(2 \\
\qquad\qquad\qquad\quad \underline{12x^2 - 34x + 24} \\
\qquad\qquad\qquad\qquad 15)30x - 45 \\
\qquad\qquad\qquad\qquad\qquad 2x - 3)6x^2 - 17x + 12(3x - 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{6x^2 - 9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -8x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-8x + 12}
\end{array}$$

Then, the H.C.F. of the expressions is $2x - 3$.

Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.

Then, the L. C. M. is $(3x - 4)(12x^2 - 4x - 21)$.

EXERCISE 25

Find the H. C. F. and L. C. M. of the following:

1. $2a^2 + a - 6, \quad 4a^2 - 8a + 3.$
2. $6x^2 - 17x + 10, \quad 9x^2 - 14x - 8.$
3. $x^2 - 6x - 27, \quad x^3 - 2x^2 - 8x + 21.$
4. $6x^2 - 31xy + 18y^2, \quad 9x^2 + 15xy - 14y^2.$
5. $8x^2 + 6x - 9, \quad 6x^3 + 7x^2 - 7x - 6.$
6. $4x^2 - 11x - 3, \quad 8x^4 + 6x^3 - 11x^2 - 23x - 5.$
7. $m^5 - 4m^3 + m^2 - 4, \quad m^4 - 2m^3 - m^2 + m + 2.$
8. $12p^2 - 19pq - 21q^2, \quad 12p^3 + 5p^2q - 11pq^2 - 6q^3.$
9. $c^4 + 7c^3 + 12c^2, \quad c^3 + 4c^2 - 9c - 36, \quad 3c^3 + 10c^2 - 15c - 28.$

10. $8x^3+27,\ 4x^3-8x^2-9x+18,\ 2x^3+x^2-11x-12.$

11. $81-x^4,\ x^4-4x^3+4x^2-4x+3.$

12. $x^4+4,\ ax^2+2ax+2a,\ x^3+3x^2+4x+2.$

13. $16c^4+8c^2+81,\ 4c^3+4c^2+c-9.$

14. $a^3+7a^2-9a-63,\ a^3+6a^2+11a+6.$

15. $(5a-3b)^2-(a+b)^2,\ 72a^2-48ab+8b^2.$

16. $a^3+6a^2x+12ax^2+8x^3,\ 4a^5+8a^4x-a^3x^2-2a^2x^3.$

V. FRACTIONS

119. A **Fraction** is an indicated quotient written usually in the form $\frac{a}{b}$, where a is the dividend, and is called the numerator, and b the divisor, and called the denominator.

120. If the same factor be introduced into, or removed from, both dividend and divisor, the quotient is not changed. Upon this principle depends the reduction of fractions to either higher or lower terms. The laws of sign for fractions are those of ordinary division. The sign before the fraction denotes whether the quotient is to be added or subtracted.

REDUCTION OF FRACTIONS

121. Change of sign,

$$+\frac{+a}{+b}=-\frac{-a}{+b}=-\frac{+a}{-b}=+\frac{-a}{-b}.$$

EXERCISE 26

Write each of the following in three other ways without changing its value:

1. $\frac{a}{2}$. 2. $\frac{n+3}{7}$. 3. $\frac{8}{2-x}$. 4. $\frac{2x-7}{x+2}$. 5. $\frac{6x-5}{(x-3)(y+4)}$.

6. $\frac{b^2-a^2}{2b^2-a^2}$. 7. $\frac{(4x-3y)(y-3x)}{(2y+x)(x-y)}$.

122. Reduction to Lowest Terms. This is accomplished by removing every factor common to both numerator and denominator. If numerator and denominator are not prime to each other, it is possible generally to factor them by inspection. When, however, the factors cannot be readily seen, the method of § 117, known as the Euclidean method, may be used.

EXERCISE 27

Reduce the following to lowest terms:

1. $\dfrac{27x^3+8}{9x^2+12x+4}$.

2. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

3. $\dfrac{12z^2+16zy-3y^2}{10z^2+zy-21y^2}$.

4. $\dfrac{2x^3+5x^2-2x+3}{6x^3-7x^2+5x-2}$.

5. $\dfrac{x^3-x^2-4x-6}{x^3+7x^2+12x+10}$.

6. $\dfrac{16x^2+16x-32}{14x^2+14x-28}$.

Simplify the following:

7. $\left(\dfrac{a^2-3a+2}{a^2+5a+4}\right)\left(1+\dfrac{4a}{a^2-2a+1}\right)$.*

8. $\dfrac{x^2-y^2}{x^2+xy}+\dfrac{y}{x}-\dfrac{2x^3y}{x^2+y^2}$.

9. $\dfrac{a^2-15a+56}{c^3-125}\times\dfrac{c^2-3c-10}{c^2-8c+16}\div\dfrac{ac+2a-14-7c}{2c-8-4a+ac}$.

10. $\dfrac{1}{x+2}+\dfrac{1}{x-2}+\dfrac{2x}{x^2+4}+\dfrac{4x^3}{x^4+16}$.

11. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}$.

* To simplify this and following examples of Exercise 27, perform the indicated operations, then reduce the resulting fraction to its lowest terms.

12. $\dfrac{bc}{(a-b)(a-c)}-\dfrac{ab}{(c-a)(b-c)}-\dfrac{ac}{(c-b)(b-a)}.$

13. $\dfrac{4}{a-1}+\dfrac{5a^2-7a-6}{6a^2-a-12}\cdot\dfrac{8a^2-16a+6}{2a^2-5a+2}.$

14. $\dfrac{4}{3x-\dfrac{x-y}{1-\dfrac{2x-3y}{3x-4y}}}.$

15. $\dfrac{m-2}{m+5}-\dfrac{4-m}{3-m}+\dfrac{1-m}{m-5}.$

16. $\left(3a+5-\dfrac{a+7}{a+2}\right)\div\left(\dfrac{5a}{a^2-10a-24}+1\right).$

17. $\left[\dfrac{1}{3x+2y}+\dfrac{1}{3x-2y}\right]\div\left[\dfrac{1}{27x^3+8y^3}+\dfrac{1}{27x^3-8y^3}\right].$

18. $\dfrac{4c^4-29c^2+25}{c^6-1}\div\left(\dfrac{4c^2-20c+25}{3c^2+3c+3}\times\dfrac{6c^2+11c-10}{9c^2-4}\right).$

19. $\dfrac{x-5+\dfrac{9(x^2+5x+6)}{x^3+6x^2+11x+6}}{\dfrac{x^4-2x^3-8x+16}{x^3+3x^2+3x+1}}.$

20. $\dfrac{\dfrac{1}{x}+\dfrac{2}{y}}{1+\dfrac{4xy+y^2}{4x^2}}-\dfrac{\dfrac{4x}{y}-2+\dfrac{y}{x}}{8x^3+y^3}.$

21. $\dfrac{x+4}{x+2}-\dfrac{x-1}{x-3}+\dfrac{x+2}{x-5}-\dfrac{x^2-x-16}{x^2-8x+15}.$

22. $\left(1-\dfrac{x^2+2x-11}{x^2+5x-14}\right)\div\dfrac{9}{x^3+343}.$

23. $\dfrac{2}{x+1}-\dfrac{x^2+3x+2}{x^2-1}\cdot\dfrac{4}{x^2+5x+6}+\dfrac{x^2+1}{x}\div\dfrac{x^4-1}{12x}.$

24. $\dfrac{(2x^2-2xy-2x)(x^2-y^2)}{[(x-y)^2-1]\div\dfrac{x}{x+y}}.$

25. $\dfrac{a^5+b^5}{a^2+b^2}\times\dfrac{a^4-b^4}{a^2b+ab^2}\times\dfrac{3a^2}{a^4-a^3b+a^2b^2-ab^3+b^4}.$

123. Under certain conditions a fraction may assume a form the value of which is not readily seen. Such forms usually occur in limiting values of fractions in which the unknown or unknowns are considered variable.

124. A *variable number*, or simply a *variable*, is a number which may assume, under the conditions imposed upon it, an indefinitely great number of different values.

A *constant* is a number which remains unchanged throughout the same discussion.

125. A *limit* of a variable is a constant number, the difference between which and the variable may be made less than any assigned number, however small.

Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, halfway between A and B; then to D, halfway between C and B; then to E, halfway between D and B; and so on indefinitely.

In this case, the distance between the moving point and B can be made less than any assigned number, however small.

Hence, the distance from A to the moving point is a variable which approaches the constant value AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

126. Interpretation of $\frac{a}{0}$.

Consider the series of fractions $\dfrac{a}{3}, \dfrac{a}{.3}, \dfrac{a}{.03}, \dfrac{a}{.003}, \cdots.$

Here each denominator after the first is one-tenth of the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made less than any assigned number, however small, and the value of the fraction greater than any assigned number, however great.

In other words,

If the numerator of a fraction remains constant, while the denominator approaches the limit 0, the value of the fraction increases without limit.

It is customary to express this principle as follows:

$$\frac{a}{0} = \infty.$$

The symbol ∞ is called *Infinity*; it simply stands for that which is greater than any number, however great, and has no fixed value.

127. Interpretation of $\frac{a}{\infty}$.

Consider the series of fractions $\frac{a}{3}, \frac{a}{30}, \frac{a}{300}, \frac{a}{3000}, \cdots$.

Here each denominator after the first is ten times the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made greater than any assigned number, however great, and the value of the fraction less than any assigned number, however small.

In other words,

If the numerator of a fraction remains constant, while the denominator increases without limit, the value of the fraction approaches the limit 0.

It is customary to express this principle as follows:

$$\frac{a}{\infty} = 0.$$

128. No *literal meaning* can be attached to such results as

$$\frac{a}{0} = \infty, \text{ or } \frac{a}{\infty} = 0;$$

for there can be no such thing as division unless the divisor is a *finite number.*

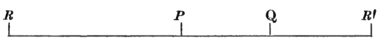
If such forms occur in mathematical investigations, they must be interpreted as indicated in §§ 126 and 127. (Compare § 86.)

THE PROBLEM OF THE COURIERS

129. The following discussion will further illustrate the form $\frac{a}{0}$, besides furnishing an interpretation of the form $\frac{0}{0}$.

The Problem of the Couriers.

Two couriers, A and B, are travelling along the same road in the same direction, RR', at the rates of m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P, and B is a miles beyond him at Q, after how many hours, and how many miles beyond P, are they together?



Let A and B meet x hours after 12 o'clock, and y miles beyond P.

They will then meet $y - a$ miles beyond Q.

Since A travels mx miles, and B nx miles, in x hours, we have

$$\begin{cases} y = mx, \\ y - a = nx. \end{cases}$$

Solving these equations, we obtain

$$x = \frac{a}{m-n}, \text{ and } y = \frac{am}{m-n}.$$

We will now discuss these results under different hypotheses.

1. $m > n$.

In this case, the values of x and y are *positive.*

This means that the couriers meet at some time *after* 12, at some point to the *right* of P.

This agrees with the hypothesis made; for if m is greater than n, A is travelling faster than B; and he must overtake him at some point beyond their positions at 12 o'clock.

2. $m < n$.

In this case, the values of x and y are *negative*.

This means that the couriers met at some time *before* 12, at some point to the *left* of P.

This agrees with the hypothesis made; for if m is less than n, A is travelling more slowly than B; and they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $a = 0$, and $m > n$ or $m < n$.

In this case, $x = 0$ and $y = 0$.

This means that the travellers are together at 12 o'clock, at the point P.

This agrees with the hypothesis made; for if $a = 0$, and m and n are unequal, the couriers are together at 12 o'clock, and are travelling at unequal rates; and they could not have been together before 12, and will not be together afterwards.

4. $m = n$, and a not equal to 0.

In this case, the values of x and y take the forms $\frac{a}{0}$ and $\frac{am}{0}$, respectively.

If $m - n$ approaches the limit 0, the values of x and y increase without limit (§ 126); hence, if $m = n$, no fixed values can be assigned to x and y, and the problem is impossible.

In this case, *the result in the form* $\frac{a}{0}$ *indicates that the given problem is impossible.*

This agrees with the hypothesis made; for if $m = n$, and a is not zero, the couriers are a miles apart at 12 o'clock, and are travelling at the same rate; and they never could have been, and never will be together.

5. $m = n$, and $a = 0$.

In this case, the values of x and y take the form $\frac{0}{0}$.

If $a = 0$, and $m = n$, the couriers are together at 12 o'clock, and travelling at the same rate.

Hence, they always have been, and always will be, together.

In this case, the number of solutions is indefinitely great; for any value of x whatever, together with the corresponding value of y, will satisfy the given conditions.

In this case, *the result in the form* $\frac{0}{0}$ *indicates that the number of solutions is indefinitely great.*

Such form is called **Indeterminate**.

130. In § 129, we found that the form $\frac{0}{0}$ indicated an expression which could have *any value whatever;* but this is not always the case.

Consider, for example, the fraction $\frac{x^2 - a^2}{x^2 - ax}$.

If $x = a$, the fraction takes the form $\frac{0}{0}$.

Now, $$\frac{x^2 - a^2}{x^2 - ax} = \frac{(x+a)(x-a)}{x(x-a)} = \frac{x+a}{x};$$

which last expression is equal to the given fraction provided x does not equal a.

The fraction $\frac{x+a}{x}$ approaches the limit $\frac{a+a}{a}$, or 2, when x approaches the limit a.

This limit we call *the value of the given fraction when* $x = a$.

Then, the value of the given fraction when $x = a$ is 2.

In any similar case, we cancel the factor which equals 0 for the given value of x, and find the limit approached by the result when x approaches the given value as a limit.

EXERCISE 28

Find the values of the following:

1. $\frac{2ax - 4a^2}{x^2 - 4a^2}$ when $x = 2a$.

2. $\frac{2x^3 - 5x^2}{4x^2 + 3x}$ when $x = 0$.

3. $\frac{x^2 - 16}{x^2 + 2x - 8}$ when $x = -4$.

4. $\frac{4x^2 - 4x - 3}{6x^2 - 17x + 12}$ when $x = \frac{3}{2}$.

5. $\frac{x^3 + 6x^2 + 12x + 8}{x^4 - 8x^2 + 16}$ when $x = -2$.

6. $\frac{x^3 - 3x^2 + 3x - 2}{x^3 - 7x + 6}$ when $x = 2$.

131. Other Indeterminate Forms.

Expressions taking the forms $\frac{\infty}{\infty}$, $0 \times \infty$, or $\infty - \infty$, for certain values of the letters involved, are also indeterminate.

1. Find the value of $(x^3+8)\left(1+\frac{1}{x+2}\right)$ when $x=-2$.

This expression takes the form $0 \times \infty$, when $x = -2$ (§ 126).

Now, $(x^3+8)\left(1+\frac{1}{x+2}\right) = x^3+8+\frac{x^3+8}{x+2}$

$$= x^3+8+x^2-2x+4 = x^3+x^2-2x+12.$$

The latter expression approaches the limit $-8+4+4+12$, or 12, when x approaches the limit -2.

This limit we call *the value of the expression when* $x = -2$; then, the value of the expression when $x = -2$, is 12.

In any similar case, we simplify as much as possible before finding the limit.

2. Find the value of $\frac{1}{1-x} - \frac{2x}{1-x^2}$ when $x = 1$.

The expression takes the form $\infty - \infty$, when $x = 1$ (§ 126).

Now, $$\frac{1}{1-x} - \frac{2x}{1-x^2} = \frac{1+x-2x}{1-x^2} = \frac{1-x}{1-x^2} = \frac{1}{1+x}.$$

The latter expression approaches the limit $\frac{1}{2}$ when x approaches the limit 1.

Then, the value of the expression when $x = 1$, is $\frac{1}{2}$.

132. Another example in which the result is indeterminate is the following:

Ex. Find the limit approached by the fraction $\frac{1+2x}{2-5x}$ when x is indefinitely increased.

Both numerator and denominator increase indefinitely in absolute value when x is indefinitely increased.

Dividing each term of the fraction by x, $$\frac{1+2x}{2-5x} = \frac{\frac{1}{x}+2}{\frac{2}{x}-5}.$$

The latter expression approaches the limit $\frac{0+2}{0-5}$ (§ 127), or $-\frac{2}{5}$, when x is indefinitely increased.

In any similar case, we divide both numerator and denominator of the fraction by the highest power of x.

EXERCISE 29

Find the limits approached by the following when x is indefinitely increased:

1. $\dfrac{4+5x-3x^2}{7-x+4x^2}$. 2. $\dfrac{2x+1}{3x^2-2}$. 3. $\dfrac{x^3-2x-4}{x^2+5x+3}$.

Find the values of the following:

4. $\dfrac{1}{x-2}-\dfrac{12}{x^3-8}$ when $x=2$.

5. $(2x^2-5x-3)\left(2+\dfrac{1}{x-3}\right)$ when $x=3$.

RATIO AND PROPORTION

RATIO

133. The **Ratio** of one number a to another number b is the quotient of a divided by b.

Thus, the ratio of a to b is $\frac{a}{b}$; it is also expressed $a:b$.

The ratios here spoken of are but fractions under another name, and *have all the properties of fractions.*

In the ratio $a:b$, a is called the *first term,* or *antecedent,* and b the *second term,* or *consequent.*

If a and b are positive numbers, and $a>b$, $\frac{a}{b}$ is called a *ratio of greater inequality;* if $a<b$, it is called a *ratio of less inequality.*

134. A ratio of greater inequality is decreased, and one of less inequality is increased, by adding the same positive number to each of its terms.

Let a and b be positive numbers, a being $>b$, and x a positive number.

Since $a>b$, $ax>bx$. (§ 59)

Adding ab to both members (§ 56),

$$ab+ax>ab+bx, \text{ or } a(b+x)>b(a+x).$$

Dividing both members by $b(b+x)$, we have

$$\frac{a}{b} > \frac{a+x}{b+x}. \qquad (\S\ 59)$$

In like manner, if $a < b$, $\frac{a}{b} < \frac{a+x}{b+x}.$

PROPORTION

135. A **Proportion** is an equation whose members are equal ratios.

Thus, if $a:b$ and $c:d$ are equal ratios,

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d},$$

is a proportion. The latter form is preferable.

136. In the proportion $a:b=c:d$, a is called the *first term*, b the *second*, c the *third*, and d the *fourth*.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

The first and fourth terms are called the *extremes*, and the second and third terms the *means*.

137. If the means of a proportion are equal, either mean is called the **Mean Proportional** between the first and last terms, and the last term is called the **Third Proportional** to the first and second terms.

Thus, in the proportion $a:b=b:c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

The **Fourth Proportional** to three numbers is the fourth term of a proportion whose first three terms are the three numbers taken in their order.

Thus, in the proportion $a:b=c:d$, d is the fourth proportional to a, b, and c.

138. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the next antecedent; as,

$$a:b=b:c=c:d=d:e.$$

PROPERTIES OF PROPORTIONS

139. In any proportion, the product of the extremes is equal to the product of the means.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Clearing of fractions, $ad = bc$.

140. From the equation $ad = bc$ (§ 139), we obtain

$$a=\frac{bc}{d},\ b=\frac{ad}{c},\ c=\frac{ad}{b},\ \text{and } d=\frac{bc}{a}.$$

That is, **in any proportion, either extreme equals the product of the means divided by the other extreme; and either mean equals the product of the extremes divided by the other mean.**

141. (Converse of § 139.) **If the product of two numbers be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.**

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$.

In like manner, we may prove that

$$\frac{a}{c}=\frac{b}{d},$$

$$\frac{c}{d}=\frac{a}{b},\ \text{etc.}$$

142. In any proportion, the terms are in proportion by ***Alternation;*** that is, the means may be interchanged.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Then, by § 139, $ad = bc$.

Then, by § 141, $\frac{a}{c}=\frac{b}{d}$.

In like manner it may be proved that the extremes can be interchanged.

143. **In any proportion, the terms are in proportion by *Inversion;* that is, the second term is to the first as the fourth term is to the third.**

Let the proportion be $\frac{a}{b} = \frac{c}{d}$,

Then, by § 139, $ad = bc$.

Whence, by § 141, $\frac{b}{a} = \frac{d}{c}$.

It follows from § 143 that, in any proportion, the means can be written as the extremes, and the extremes as the means.

144. **The mean proportional between two numbers is equal to the square root of their product.**

Let the proportion be $\frac{a}{b} = \frac{b}{c}$.

Then, by § 139, $b^2 = ac$, or $b = \sqrt{ac}$.

145. **In any proportion, the terms are in proportion by *Composition;* that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.**

Let the proportion be $\frac{a}{b} = \frac{c}{d}$.

Then, $ad = bc$.

Adding each member of the equation to ac,

$$ac + ad = ac + bc, \text{ or } a(c + d) = c(a + b).$$

By § 141, $\frac{a+b}{a} = \frac{c+d}{c}$.

We may also prove $\frac{a+b}{b} = \frac{c+d}{d}$.

146. In like manner we may also prove that the terms of any proportion are in proportion by ***Division;*** **that is, the difference between the first two terms is to the first term as the difference between the last two terms is to the third term.**

The proof is left to the student.

147. In any proportion, the terms are in proportion by *Composition and Division*; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

The proof is left to the student. HINT. — Divide the result of § 145 by that of § 146.

148. In any proportion, if the first two terms be multiplied by any number, as also the last two, the resulting numbers will be in proportion.

Let the proportion be $\frac{a}{b} = \frac{c}{d}$; then, $\frac{ma}{mb} = \frac{nc}{nd}$.

(Either m or n may be unity; that is, the terms of either ratio may be multiplied without multiplying the terms of the other.)

149. In any proportion, if the first and third terms be multiplied by any number, as also the second and fourth terms, the resulting numbers will be in proportion.

Let the proportion be $\frac{a}{b} = \frac{c}{d}$; then, $\frac{ma}{nb} = \frac{mc}{nd}$.

(Either m or n may be unity.)

150. In any number of proportions, the products of the corresponding terms are in proportion.

Let the proportions be $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$.

Multiplying, $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, or $\frac{ae}{bf} = \frac{cg}{dh}$.

In like manner, the theorem may be proved for any number of proportions.

151. In any proportion, like powers or like roots of the terms are in proportion.

Let the proportion be $\frac{a}{b} = \frac{c}{d}$; then, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$.

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

152. In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $a:b=c:d=e:f.$

Then, by § 139, $ad=bc,$

and $af=be.$

Also, $ab=ba.$

Adding, $a(b+d+f)=b(a+c+e).$

Whence, $a:b=a+c+e:b+d+f.$ (§ 141)

In like manner, the theorem may be proved for any number of equal ratios.

153. If three numbers are in continued proportion, the first is to the third as the square of the first is to the square of the second.

Let the proportion be $a:b=b:c$; or $\frac{a}{b}=\frac{b}{c}.$

Then, $\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{c}=\frac{a^2}{b^2}.$

154. If four numbers are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.

Let the proportion be $a:b=b:c=c:d$; or $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}.$

Then, $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{d}=\frac{a^3}{b^3}.$

Similarly, it may be shown that if n numbers are in continued proportion, the first antecedent is to the last consequent as the nth power of the first antecedent is to the nth power of its consequent.

155. Examples.

1. If $x:y=(x+z)^2:(y+z)^2$, prove z the mean proportional between x and y.

From the given proportion, by § 139,

$$y(x+z)^2=x(y+z)^2.$$

Or, $x^2y+2xyz+yz^2=xy^2+2xyz+xz^2.$

Transposing, $x^2y-xy^2=xz^2-yz^2.$

Dividing by $x-y$, $xy=z^2.$

Therefore, z is the mean proportional between x and y (§ 144).

The theorem of § 147 saves work in the solution of a certain class of fractional equations.

2. Solve the equation $\dfrac{2x+3}{2x-3}=\dfrac{2b-a}{2b+a}$.

Regarding this as a proportion, we have by composition and division,

$$\frac{4x}{6}=\frac{4b}{-2a},\text{ or }\frac{2x}{3}=-\frac{2b}{a};\text{ whence, }x=-\frac{3b}{a}.$$

3. Prove that if $\dfrac{a}{b}=\dfrac{c}{d}$, then

$$a^2-b^2:a^2-3ab=c^2-d^2:c^2-3cd.$$

Let $\dfrac{a}{b}=\dfrac{c}{d}=x$, whence, $a=bx$; then,

$$\frac{a^2-b^2}{a^2-3ab}=\frac{b^2x^2-b^2}{b^2x^2-3b^2x}=\frac{x^2-1}{x^2-3x}=\frac{\dfrac{c^2}{d^2}-1}{\dfrac{c^2}{d^2}-\dfrac{3c}{d}}=\frac{c^2-d^2}{c^2-3cd}.$$

Then, $a^2-b^2:a^2-3ab=c^2-d^2:c^2-3cd.$

EXERCISE 30

1. Find the mean proportional between .0289 and 1.69.
2. Find the mean proportional between $1\frac{75}{121}$ and $12\frac{24}{25}$.
3. Find the third proportional to $\frac{15}{16}$ and $1\frac{1}{8}$.
4. Find the fourth proportional to $9\frac{9}{80}$, $16\frac{1}{5}$, and $\frac{9}{16}$.
5. Find the fourth proportional to m, n, and r.
6. Write in the form of a proportion: $x^2-2x-15=a^2$.

Solve, using composition and division:

7. $\dfrac{4x+5}{4x-5}=\dfrac{x+5}{x-3}$.

8. $\dfrac{x+a}{x-a}=\dfrac{b+c}{b-c}$.

10. $\dfrac{2x+7}{2x-3}=\dfrac{5x+1}{5x-9}$.

9. $\dfrac{a^{\frac{1}{2}}-x}{a^{\frac{1}{2}}+x}=\dfrac{1}{3}$.

11. $\dfrac{(m+1)^{\frac{1}{2}}+(m-1)^{\frac{1}{2}}}{(m+1)^{\frac{1}{2}}-(m-1)^{\frac{1}{2}}}=3$.

12. If $\dfrac{a}{b}=\dfrac{b}{c}$, show that $a:c=b^2:c^2$.

13. If $a : b = c : d$, show that $\frac{4a^2c - 6abc + 9b^2c}{2a^2 - 3ab} = \frac{8c^3 + 27d^3}{4c^2 - 9d^2}$.

14. Find two numbers in the ratio of $2 : 3$, such that the sum of their squares shall be 208.

15. Find two numbers in the ratio of $3 : 1$, such that the difference of their squares is 200.

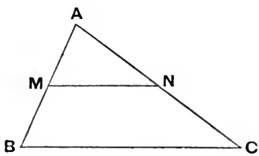
16. Two numbers are in the ratio of $5 : 7$. If 6 be added to each, they will be in the ratio of $7 : 9$. Find the numbers.

17. Two numbers are in the ratio of $2 : 5$. If 4 be added to each number, the resulting ratio will be twice the ratio had 4 been subtracted from each number. Find the numbers.

18. The difference between two numbers is 6, and the difference between their squares is 60. What is the ratio of their sum to their difference?

19. In similar figures in geometry, homologous sides are proportional. If a pole 30 feet high casts a shadow 42 feet long, how high must a pole be to cast a shadow 35 feet long?

20. A ladder 40 feet long leans against the side of a building, with its foot 12 feet from the building. A second ladder, $40\frac{1}{2}$ feet long, makes the same angle with the building as the first ladder. How far is the foot of the second ladder from the building?



21. In the triangle ABC, MN is drawn parallel to BC and divides the other two sides proportionally. If $AM = 12$, $\frac{AM}{AN} = \frac{2}{3}$, and $BC = 48$, how long is AC? (M is the middle point of AB.) What is the ratio of AN to MN?

22. The areas of any two similar figures are to each other as the squares of their homologous parts. If a regular hexagon has a side equal to 6 and an area of $54\sqrt{3}$, what is the area of a regular hexagon whose side is 2?

23. The area of a circle is $6\frac{1}{4}$ times that of another circle. If the radius of the first circle is 5, what is the radius of the second circle?

24. If the altitude of a triangle is twice that of a similar triangle, how do their areas compare?

25. The volume of a rectangular solid is equal to the product of its three dimensions, x, y, and z. If $xyz = v$ and $x : y : z = a : b : c$, find x, y, and z in terms of a, b, c, and v.

26. Find three numbers in continued proportion whose sum is 63, the second being 4 times the first.

27. Given the proportion $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, where $d = 81$ and $\frac{a}{b} = \frac{1}{3}$. Find a, b, and c.

28. If $2a - 3b : 4a - 5b = 2b - 3c : 4b - 5c$, prove b is the mean proportional to a and c.

29. If $3a + 5b : 4a - 7b = 3c + 5d : 4c - 7d$, prove $\frac{a}{b} = \frac{c}{d}$.

30. Find two numbers in the ratio of a to b, such that if each be increased by $\frac{c}{d}$ they will be in the ratio of e to f.

31. $\frac{4x+7}{5} - \frac{8x+4}{15} - \frac{12x+1}{45} = \frac{5x-1}{9(5x+2)}$. Solve for x.

32. $\frac{a+b}{x} + \frac{a-2b}{x+a} = \frac{(2a-b)x+3ab}{x^2-a^2}$. Solve for x.

33. A man borrows a certain sum, paying interest at the rate of 5%. After repaying \$180, his interest rate on the balance is reduced to $4\frac{1}{4}$%, and his annual interest is now less by \$10.80. Find the sum borrowed.

34. The digits of a certain number are three consecutive numbers, of which the middle digit is the greatest, and the first digit the least. If the number be divided by the sum of its digits, the quotient is $\frac{229}{7}$. Find the number.

35. A certain number of apples were divided between three boys. The first received one-half the entire number, with one apple additional, the second received one-third the remainder, with one apple additional, and the third received the remainder, 7. How many apples were there?

36. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger train runs 112 miles. Find the rate of each train.

37. A and B each fire 40 times at a target; A's hits are one-half as numerous as B's misses, and A's misses exceed by 15 the number of B's hits. How many times does each hit the target?

38. A freight train travels from A to B at the rate of 12 miles an hour. After it has been gone $3\frac{1}{2}$ hours, an express train leaves A for B, travelling at the rate of 45 miles an hour, and reaches B 1 hour and 5 minutes ahead of the freight. Find the distance from A to B, and the time taken by the express train.

39. A tank has three taps. By the first it can be filled in 3 hours 10 minutes, by the second it can be filled in 4 hours 45 minutes, and by the third it can be emptied in 3 hours 48 minutes. How many hours will it take to fill it if all the taps are open?

40. A man invested a certain sum at $3\frac{3}{4}\%$, and $\frac{11}{15}$ this sum at $4\frac{1}{4}\%$; after paying an income tax of 5%, his net annual income is $195.70. How much did he invest in each way?

VARIATION

156. One variable number (§ 124) is said to *vary directly* as another when the ratio of any two values of the first equals the ratio of the corresponding values of the second.

It is usual to omit the word "directly" and simply say that one number *varies* as another.

Thus, if a workman receives a fixed number of dollars per diem, the number of dollars received in m days will be to the number received in n days as m is to n.

Then, the ratio of any two numbers of dollars received equals the ratio of the corresponding numbers of days worked.

Hence, the number of dollars which the workman receives *varies* as the number of days during which he works.

157. The symbol $\propto$ is read "*varies as*"; thus, $a \propto b$ is read "a varies as b."

158. One variable number is said to *vary inversely* as another when the first varies directly as the *reciprocal* of the second.

Thus, the number of hours in which a railway train will traverse a fixed route varies inversely as the speed; if the speed be *doubled*, the train will traverse its route in *one-half* the number of hours.

159. One variable number is said to vary as two others *jointly* when it varies directly as their product.

Thus, the number of dollars received by a workman in a certain number of days varies jointly as the number which he receives in one day, and the number of days during which he works.

160. One variable number is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, the attraction of a body varies directly as the amount of matter, and inversely as the square of the distance.

161. *If $x \propto y$, then x equals y multiplied by a constant number.*

Let x' and y' denote a *fixed* pair of corresponding values of x and y, and x and y any other pair.

By the definition of § 156, $\frac{x}{y} = \frac{x'}{y'}$; or, $x = \frac{x'}{y'}y$.

Denoting the constant ratio $\frac{x'}{y'}$ by m, we have

$$x = my.$$

162. It follows from §§ 158, 159, 160, and 161 that:

1. *If x varies inversely as y,* $x = \frac{m}{y}$.

2. *If x varies jointly as y and z,* $x = myz$.

3. *If x varies directly as y and inversely as z,* $x = \frac{my}{z}$.

163. *If $x \propto y$, and $y \propto z$, then $x \propto z$.*

By § 161, if $x \propto y$, $\qquad x = my. \qquad (1)$

And if $y \propto z$, $\qquad y = nz.$

Substituting in (1), $\qquad x = mnz.$

Whence, by § 161, $\qquad x \propto z.$

164. *If $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when both y and z vary.*

Let y' and z' be the values of y and z, respectively, when x has the value x'.

Let y be changed from y' to y'', z remaining constantly equal to z', and let x be changed in consequence from x' to X.

Then, by § 156, $$\frac{x'}{X} = \frac{y'}{y''}. \qquad (1)$$

Now, let z be changed from z' to z'', y remaining constantly equal to y'', and let x be changed in consequence from X to x .

Then, $$\frac{X}{x''} = \frac{z'}{z''}. \qquad (2)$$

Multiplying (1) by (2), $$\frac{x'}{x''} = \frac{y'z'}{y''z''}. \qquad (3)$$

Now if *both* changes are made, that is, y from y' to y'' and z from z' to z'', x is changed from x' to x'', and yz is changed from $y'z'$ to $y''z''$.

Then by (3), the ratio of any two values of x equals the ratio of the *corresponding values* of yz; and, by § 156, $x \propto yz$.

The following is an illustration of the above theorem:

It is known, by Geometry, that the area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant; hence, when both base and altitude vary, the area varies as their product.

165. Problems.

Problems in variation are readily solved by converting the variation into an equation by aid of §§ 161 or 162.

1. If x varies inversely as y, and equals 9 when $y=8$, find the value of x when $y=18$.

If x varies inversely as y, $x=\frac{m}{y}$ (§ 162).

Putting $x=9$ and $y=8$, $9=\frac{m}{8}$, or $m=72$.

Then, $x=\frac{72}{y}$; and, if $y=18$, $x=\frac{72}{18}=4$.

Since variation is simply another way of stating a proportion, the problems in variation may be solved readily by means of proportion.

E.g. In the above problem

$$x \propto \frac{1}{y},$$

$$x=\frac{m}{y}.$$

This equation is true for any assigned values of the variables.

Then,
$$x_1=\frac{m}{y_1}, \qquad (1)$$

$$x_2=\frac{m}{y_2}. \qquad (2)$$

Dividing (1) by (2)
$$\frac{x_1}{x_2}=\frac{y_2}{y_1} \qquad (3)$$

which is in the form of inverse proportion. Substituting the given values of x and y in (3), we have

$$\frac{9}{x_2}=\frac{18}{8},$$

whence
$$x_2=\frac{9\cdot 8}{18}=4.$$

2. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9, respectively?

Let B, H, and A denote the base, altitude, and area, respectively, of any triangle, and B' the base of the required triangle.

Since A varies jointly as B and H, $A=mBH$ (§ 162).

Therefore, the area of the first triangle is $m\times 10\times 3$, or $30\,m$, and the area of the second is $m\times 6\times 9$, or $54\,m$.

Then, the area of the required triangle is $30\,m + 54\,m$, or $84\,m$.
But, the area of the required triangle is also $m \times B' \times 12$.

Therefore, $$12\,mB' = 84\,m, \text{ or } B' = 7.$$

Or using proportion and letting A_1 = area of first triangle, A_2 = area of second, A_3 = area of third.

$$A_3 = A_1 + A_2$$

$$A_1 = mB_1H_1. \quad (1)$$

$$A_2 = mB_2H_2. \quad (2)$$

$$A_3 = mB_3H_3. \quad (3)$$

Adding (1) and (2)

$$A_1 + A_2 = m(B_1H_1 + B_2H_2). \quad (4)$$

Dividing (4) by (3)

$$\frac{A_1 + A_2}{A_3} = \frac{m(B_1H_1 + B_2H_2)}{m(B_3H_3)},$$

or, $$1 = \frac{B_1H_1 + B_2H_2}{B_3H_3}. \quad (5)$$

Substituting the given values of B and H in (5) we have

$$1 = \frac{10 \cdot 3 + 6 \cdot 9}{12\,B_3},$$

whence, $$B_3 = 7.$$

EXERCISE 31

1. If $x \propto y$, and $x = 3$ when $y = 12$, what is the value of x when $y = 28$?

2. If $y \propto x^2$, and $y = 4$ when $x = 1$, what is the value of y in terms of x^2?

3. If y varies inversely as x, and $y = 4$ when $x = -3$, what is the value of y when $x = 2$?

4. If x varies directly as y and inversely as z, and $x = \frac{1}{2}$ when $y = \frac{2}{3}$ and $z = \frac{3}{4}$, what is the value of x when $y = \frac{4}{5}$ and $z = \frac{5}{12}$?

5. If x varies jointly as y and z and $x = -20$ when $y = 2$ and $z = 8$, what is the value of x when $y = -\frac{1}{2}$ and $z = 16$?

6. If $(3\,x + 4) \propto (2\,y - 5)$ when $x = -1$ and $y = 4$, what is the value of x when $y = 19$?

7. If x^2 varies inversely as y^3, when $x=4$ and $y=2$, what is the value of y when $x=\frac{4}{27}$?

8. If x equals the sum of two numbers, one of which varies directly as y and the other inversely as z^2, and $x=47$ when $y=-16$ and $z=2$, and $x=2$ when $y=-2$ and $z=1$, find the value of x when $y=3$ and $z=\frac{1}{3}$.

9. The area of a triangle varies jointly as its base and altitude. If the area of a triangle whose base is 6 and whose altitude is 9 is 27, what is the base of a triangle whose area is 44 and whose altitude is 11?

10. The distance through which a body falls from rest varies as the square of the time during which it falls. If a body falls 900 feet in 7.5 seconds, how many feet will it fall in 16 seconds?

11. The illumination from a source of light varies inversely as the square of the distance from the source. How far must an object 20 feet from the light be moved in order that it may receive twice as much light?

12. A circular plate of lead, 17 inches in diameter, is melted and formed into three circular plates of the same thickness. If the diameters of two of the plates are 8 and 9 inches respectively, find the diameter of the other; it being given that the area of a circle varies as the square of its diameter.

13. A cow tied to a stake by a rope 24 yards long will graze over the area within her reach in three days. She breaks her rope and, in repairing it, it is shortened $1\frac{1}{2}$ feet. In how many days will she graze over the new area?

14. A pump supplying the water for a building has a 10-inch stroke and a cylinder 4 inches in diameter. It is not possible to increase the number of strokes of the pump, nor to increase the length of the cylinder. By how much must the diameter be increased if 50% is added to the capacity of the pump? (The volumes of cylinders vary as the product of the base and altitude.)

VI. INVOLUTION AND EVOLUTION

166. We have already given (Chapter III) the involution and evolution of monomials. We will now consider involution and evolution of polynomials.

167. Square of a polynomial. By actual multiplication

$$(a+b+c)^2 = a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

In like manner

$$(a+b+c+d)^2$$
$$= a^2+b^2+c^2+d^2+2\,ab+2\,ac+2\,ad+2\,bc+2\,bd+2\,cd,$$

and so on for the square of any polynomial.

The law observed may be stated as follows:

The square of a polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each of the following terms.

Ex. Expand $(2\,x^2-3\,x-5)^2$.

The squares of the terms are $4\,x^4$, $9\,x^2$, and 25.

Twice the product of the first term by each of the following terms gives the results $-12\,x^3$ and $-20\,x^2$.

Twice the product of the second term by the following term gives the result $30\,x$.

Then, $(2\,x^2-3\,x-5)^2 = 4\,x^4+9\,x^2+25-12\,x^3-20\,x^2+30\,x$
$= 4\,x^4-12\,x^3-11\,x^2+30\,x+25.$

168. Cube of a binomial. By actual multiplication

$$(a+b)^3 = a^3+3\,a^2b+3\,ab^2+b^3.$$

That is, the cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

In like manner, the cube of the difference of two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in parenthesis, and regarded as a single term.

169. Square Root of any Polynomial Perfect Square.

By § 167, $(a+b+c)^2 = a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2$

$$= a^2 + (2\,a + b)\,b + (2\,a + 2\,b + c)c. \qquad (1)$$

Then, if the square of a trinomial be arranged in order of powers of some letter:

I. The square root of the first term gives the first term of the root, a.

II. If from (1) we subtract a^2, we have

$$(2\,a + b)\,b + (2\,a + 2\,b + c)\,c. \qquad (2)$$

The first term of this, when expanded, is $2\,ab$; if this be divided by twice the first term of the root, $2\,a$, we have the next term of the root, b.

III. If from (2) we subtract $(2\,a+b)b$, we have

$$(2\,a + 2\,b + c)\,c. \qquad (3)$$

The first term of this, when expanded, is $2\,ac$; if this be divided by twice the first term of the root, $2\,a$, we have the last term of the root, c.

IV. If from (3) we subtract $(2\,a + 2\,b + c)\,c$, there is no remainder.

Similar considerations hold with respect to the square of a polynomial of any number of terms.

170. The principles of § 169 may be used to find the square root of a polynomial perfect square of any number of terms.

Let it be required to find the square root of

$$4\,x^4 + 12\,x^3 - 7\,x^2 - 24\,x + 16.$$

$$\begin{array}{rl|l}
 & 4\,x^4 + 12\,x^3 - 7\,x^2 - 24\,x + 16 & \underline{2\,x^2 + 3\,x - 4} \\
 & a^2 = 4\,x^4 & \\ \hline
2\,a + b = 4\,x^2 + 3\,x & 12\,x^3 - 7\,x^2 - 24\,x + 16, & \text{1st Rem.} \\
3\,x & 12\,x^3 + 9\,x^2 & \\ \hline
2\,a + 2\,b + c = 4\,x^2 + 6\,x - 4 & -16\,x^2 - 24\,x + 16, & \text{2d Rem.} \\
-4 & -16\,x^2 - 24\,x + 16 &
\end{array}$$

$2\,x^2 + 3\,x - 4$ is called the square root and $2\,a$ the first trial divisor. $2\,a + b$ is the first complete divisor.

We then have the following rule for extracting the square root of a polynomial perfect square:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial divisor.

171. Cube Root of any Polynomial Perfect Cube.

By § 168, $(a+b+c)^3 = [(a+b)+c]^3$

$$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$= a^3 + (3a^2 + 3ab + b^2)b + [3(a+b)^2 + 3(a+b)c + c^2]c. \quad (1)$$

Then, if the cube of a trinomial be arranged in order of powers of some letter:

I. The cube root of the first term gives the first term of the cube root, a.

II. If from (1) we subtract a^3, we have

$$(3a^2 + 3ab + b^2)b + [3(a+b)^2 + 3(a+b)c + c^2]c. \quad (2)$$

The first term of this, when expanded, is $3a^2b$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the next term of the root, b.

III. If from (2) we subtract $(3a^2 + 3ab + b^2)b$, we have

$$[3(a+b)^2 + 3(a+b)c + c^2]c. \quad (3)$$

The first term of this, when expanded, is $3a^2c$; if this be divided by three times the square of the first term of the root, $3a^2$, we have the last term of the root, c.

IV. If from (3) we subtract $[3(a+b)^2+3(a+b)c+c^2]c$, there is no remainder.

Similar considerations hold with respect to the cube of polynomials of any number of terms.

172. The principles of § 171 may be used to find the cube root of a polynomial perfect cube of any number of terms.

Let it be required to find the cube root of

$$x^6+6x^5+3x^4-28x^3-9x^2+54x-27.$$

		x^2+2x-3
		$x^6+6x^5+3x^4-28x^3-9x^2+54x-27$
		$a^3=x^6$
$3a^2+3ab+b^2=$	$3x^4+6x^3+4x^2$	$6x^5+3x^4-28x^3-9x^2+54x-27$
	$2x$	$6x^5+12x^4+8x^3$
$3(a+b)^2=$	$3x^4+12x^3+12x^2$	$-9x^4-36x^3-9x^2+54x-27$
$3(a+b)c+c^2=$	$-9x^2-18x+9$	
	$3x^4+12x^3+3x^2-18x+9$	
	-3	$-9x^4-36x^3-9x^2+54x-27$

The first term of the root is the cube root of x^6, or x^2.

Subtracting the cube of x^2, or x^6, from the given expression, the first remainder is $6x^5+3x^4-28x^3-9x^2+54x-27$.

Dividing the first term of this by three times the square of the first term of the root, $3x^4$, we have the next term of the root, $2x$ (§ 171, II).

Now, $3ab+b^2$ equals $3\times x^2\times 2x+(2x)^2$, or $6x^3+4x^2$.

Adding this to $3x^4$, multiplying the result by $2x$, and subtracting the product, $6x^5+12x^4+8x^3$, from the first remainder, gives the second remainder, $-9x^4-36x^3-9x^2+54x-27$ (§ 171, III).

Dividing the first term of this by three times the square of the first term of the root, $3x^2$, we have the last term of the root, -3.

Now, $3(a+b)^2$ equals $3(x^2+2x)^2$, or $3x^4+12x^3+12x^2$; $3(a+b)c$ equals $3(x^2+2x)(-3)$, or $-9x^2-18x$; and $c^2=9$.

Adding these results, we have $3x^4+12x^3+3x^2-18x+9$.

Subtracting from the second remainder the product of this by -3, or $-9x^4-36x^3-9x^2+54x-27$, there is no remainder; then, x^2+2x-3 is the required root (§ 171, IV).

The expressions $3x^4$ and $3x^4+12x^3+12x^2$ are called *trial divisors*, and the expressions $3x^4+6x^3+4x^2$ and $3x^4+12x^3+3x^2-18x+9$ *complete divisors*.

We then have the following rule for finding the cube root of a polynomial perfect cube:

Arrange the expression according to the powers of some letter.

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given expression; arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the result as the next term of the root.

Add to the trial divisor three times the product of the term of the root last obtained by the part of the root previously found, and the square of the term of the root last obtained.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the *part of the root already found* for the next trial divisor.

EXERCISE 32

Find the square roots of the following:

1. $4a^4 + 12a^3b - 7a^2b^2 - 24ab^3 + 16b^4.$

2. $49m^4 - 5m^2 - 42m^3 + 1 + 6m.$

3. $9a^2 - 24ab - 36ac + 16b^2 + 48bc + 36c^2.$

4. $\frac{c^4}{4} + \frac{13c^2}{3} - 2c^3 + \frac{1}{9} - \frac{4c}{3}.$

5. $x^6 + 5x^4 + 14x^3 - 6x^5 + 1 - 4x - 2x^2.$

6. $4m + 25m^{\frac{1}{2}} - 12m^{\frac{3}{4}} + 16 - 24m^{\frac{1}{4}}.$

7. $64c^2 - 80c - 23 + 9c^{-2} + 30c^{-1}.$

8. $4x + 9y^{-4} + 4x^{\frac{1}{2}}y^{-2} + 24x^{\frac{1}{4}}y^{-3} - 16x^{\frac{3}{4}}y^{-1}.$

9. $6yz^{-2} + 4x^{-2} + y^2 - 4x^{-1}y - 12x^{-1}z^{-2} + 9z^{-4}.$

10. $4a^3 + 29a - 4a^{\frac{5}{2}} + 21a^2 - 20a^{\frac{1}{2}} + 4 - 18a^{\frac{3}{2}}$.

Find the cube roots of the following:

11. $343x^3 - 441x^2y + 189xy^2 - 27y^3$.

12. $x^6 - 9x^5 + 21x^4 + 9x^3 - 42x^2 - 36x - 8$.

13. $18a^4 - 13a^3 + 1 + 8a^6 + 9a^2 - 3a - 12a^5$.

14. $54m^5 + 44m^3 + 1 + 27m^6 + 63m^4 + 6m + 21m^2$.

15. $n^6 + 2n^4 + n^3 - \frac{2}{3}n^2 - 3n^5 - \frac{n}{3} - \frac{1}{27}$.

16. $64a^{\frac{3}{2}}b^{-6} - 240ab^{-4}c + 300a^{\frac{1}{2}}b^{-2}c^2 - 125c^3$.

17. $8s^3 + 36s^2 + 18s - 81 - 27s^{-1} + 81s^{-2} - 27s^{-3}$.

18. $21a^{\frac{1}{2}} - 54a^{\frac{5}{4}} + 27a^{\frac{3}{2}} + 63a - 44a^{\frac{3}{4}} + 1 - 6a^{\frac{1}{4}}$.

19. $x^{-3} - 3x^{-2}y^{\frac{1}{2}} + 3x^{-1}y - z^3 - 3x^{-2}z - y^{\frac{3}{2}} + 6x^{-1}y^{\frac{1}{2}}z - 3yz$
$+ 3x^{-1}z^2 - 3yz^2$.

20. $a + 6a^{\frac{2}{3}}b^{-1} + 12a^{\frac{1}{3}}b^{-2} + 8b^{-3} + 3a^{\frac{2}{3}}c^{-2} + 12a^{\frac{1}{3}}b^{-1}c^{-2}$
$+ 12b^{-2}c^{-2} + 3a^{\frac{1}{3}}c^{-4} + 6b^{-1}c^{-4} + c^{-6}$.

Find the fourth roots of the following:

21. $81a^{10} - 36a^{\frac{15}{2}}x^{-\frac{5}{3}} + 6a^5x^{-\frac{10}{3}} - \frac{4}{9}a^{\frac{5}{2}}x^{-5} + \frac{1}{81}x^{-\frac{20}{3}}$.

22. $x^8 - 12x^7 + 50x^6 - 72x^5 - 21x^4 + 72x^3 + 50x^2 + 12x + 1$.

Find the sixth roots of the following:

23. $64m^{12} - 192m^{10} + 240m^8 - 160m^6 + 60m^4 - 12m^2 + 1$.

24. $a^3 - 3a^{\frac{5}{2}}b^{\frac{3}{2}} + \frac{15}{4}a^2b^3 - \frac{5}{2}a^{\frac{3}{2}}b^{\frac{9}{2}} + \frac{15}{16}ab^6 - \frac{3}{16}a^{\frac{1}{2}}b^{\frac{15}{2}} + \frac{1}{64}b^9$.

173. Square Root of any Integral Perfect Square.

The square root of an integral perfect square may be found in the same way as the square root of a polynomial.

We have the following rule for finding the square root of an integral perfect square:

Separate the number into periods of two digits each, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first digit of the root; subtract the square of the first root digit from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last digit, by twice the part of the root already found, and annex the quotient to the root, and also to the trial divisor.

Multiply the complete divisor by the root digit last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial divisor.

Note 1: It sometimes happens that, on multiplying a complete divisor by the digit of the root last obtained, the product is greater than the remainder.

In such a case, the digit of the root last obtained is too great, and one less must be substituted for it.

Note 2: If any root digit is 0, annex 0 to the trial divisor, and annex to the remainder the next period.

Ex. Required the square root of 15376.

$$
\begin{array}{rr|l}
 & 1'53'76 & 100 + 20 + 4 \\
 & a^2 = \underline{1\ 00\ 00} & = a + b + c \\
2a + b = 200 + 20 & 53\ 76 & \\
b = \underline{20} & \underline{44\ 00} & = (2a + b)b \\
2a + 2b + c = 200 + 40 + 4 & 9\ 76 & \\
\underline{4} & \underline{9\ 76} & = (2a + 2b + c)c
\end{array}
$$

Pointing the number in accordance with the rule of § 173, we find that there are three digits in its square root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose square is less than 15376; this we find to be 100.

Subtracting a^2, or 10000, from the given number, the result is 5376.

Dividing the remainder by $2a$, or 200, we have the quotient 26+; which suggests that b equals 20.

Adding this to $2a$, or 200, and multiplying the result by b, or 20, we have 4400; which, subtracted from 5376, leaves 976.

Since this remainder equals $(2a + 2b + c)c$, we can get c approximately by dividing it by $2a + 2b$, or $200 + 40$.

Dividing 976 by 240, we have the quotient 4+; which suggests that c equals 4.

Adding this to 240, multiplying the result by 4, and subtracting the product, 976, there is no remainder.

Then 124 is the square root.

Omitting the ciphers for the sake of brevity, and condensing the operation, we may arrange the work of the example as follows:

```
     1'53'76 | 124
     1
    ------
  22 | 53
     | 44
    ------
 244 | 976
     | 976
```

CUBE ROOT OF AN ARITHMETICAL NUMBER

174. The cube root of 1000 is 10; of 1000000 is 100, etc.

Hence, the cube root of a number between 1 and 1000 is between 1 and 10; the cube root of a number between 1000 and 1000000 is between 10 and 100; etc.

That is, the integral part of the cube root of an integer of one, two, or three digits contains *one* digit; of an integer of four, five, or six digits contains *two* digits; and so on.

Hence, **if a point be placed over every third digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its cube root.**

175. Cube Root of any Integral Perfect Cube.

The cube root of an integral perfect cube may be found in the same way as the cube root of a polynomial.

Required the cube root of 12487168.

$$
\begin{array}{rr|r|l}
 & & 12487168 & 200 + 30 + 2 \\
 & & a^3 = 8000000 & = a + b + c \\
\hline
3\,a^2 = & 120000 & 4487168 & \\
3\,ab = & 18000 & & \\
b^2 = & 900 & & \\
\hline
 & 138900 & & \\
 & 30 & 4167000 & \\
\hline
3(a+b)^2 = & 158700 & 320168 & \\
3(a+b)c = & 1380 & & \\
c^2 = & 4 & & \\
\hline
 & 160084 & & \\
 & 2 & 320168 & \\
\hline
\end{array}
$$

Pointing the number in accordance with the rule of § 174, we find that there are three digits in the cube root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose cube is less than 12487168; this we find to be 200.

Subtracting a^3, or 8000000, from the given number, the result is 4487168.

Dividing this by $3\,a^2$, or 120000, we have the quotient 37+; which suggests that b equals 30.

Adding to the divisor 120000, $3\,ab$, or 18000, and b^2, or 900, we have 138900.

Multiplying this by b, or 30, and subtracting the product 4167000 from 4487168, we have 320168.

Since this remainder equals $[3(a+b)^2 + 3(a+b)c + c^2]c$ (§ 171, III), we can get c approximately by dividing it by $3(a+b)^2$, or 158700.

Dividing 320168 by 158700, the quotient is 2+; which suggests that c equals 2.

Adding to the divisor 158700, $3(a+b)c$, or 1380, and c^2, or 4, we have 160084; multiplying this by 2, and subtracting the product, 320168, there is no remainder.

Then, 200 + 30 + 2, or 232, is the required cube root.

176. Omitting the ciphers for the sake of brevity, and condensing the process, the work of the example of § 175 will stand as follows:

	12487168	232
	8	
1200	4487	
180		
9		
1389	4167	
158700	320168	
1380		
4		
160084	320168	

The numbers 120000 and 158700 are called *trial divisors*, and the numbers 138900 and 160084 are called *complete divisors*.

We then have the following rule for finding the cube root of an integral perfect cube:

Separate the number into periods by pointing every third digit, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first digit of the root; subtract the cube of the first root digit from the left-hand period, and to the result annex the next period.

Divide this remainder by three times the square of the part of the root already found, with two ciphers annexed, and write the quotient as the next digit of the root.

Add to the trial divisor three times the product of the last root digit by the part of the root previously found, with one cipher annexed, and the square of the last root digit.

Multiply the complete divisor by the digit of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the part of the root already found, with two ciphers annexed, for the next trial divisor.

Note 1: Note 1, § 173, applies with equal force to the above rule.

Note 2: If any root-figure is 0, annex two ciphers to the trial divisor, and annex to the remainder the next period.

177. In the example of § 175, the first complete divisor is

$$3\,a^2 + 3\,ab + b^2. \qquad (1)$$

The next trial divisor is $3(a+b)^2$, or $3\,a^2 + 6\,ab + 3\,b^2$.

This may be obtained from (1) by adding to it its second term, and double its third term.

That is, **if the first number and the double of the second number required to complete any trial divisor be added to the complete divisor, the result, with two ciphers annexed, will give the next trial divisor.**

This rule saves much labor in forming the trial divisors.

Ex. Find the cube root of 157464.

	157464	54
	125	
7500	32464	
600		
16		
8116	32464	

EXERCISE 33

Find the square roots of the following:

1. 5294601.
2. 68.7241.
3. .00098596.
4. 567762.25.
5. .0037319881.

Find the cube roots of the following:

6. 658503.
7. 1953125.
8. 748.613312.
9. .000070444997.
10. .000001601613.

Find the first four figures of the square roots of:

11. 3.
12. $\frac{7}{8}$.
13. $\frac{15}{16}$.
14. $\frac{2}{7}$.
15. $\frac{2075}{3072}$.

Find the first four figures of the cube roots of:

16. 5.
17. 16.
18. $\frac{1}{4}$.
19. .27.
20. $\frac{3}{40}$.

OTHER POWERS

178. A **Series** is a succession of terms.

A **Finite Series** is one having a limited number of terms.

An **Infinite Series** is one having an unlimited number of terms.

179. In §§ 103 and 168 we gave rules for finding the square or cube of any binomial.

The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

180. Proof of the Binomial Theorem for a Positive Integral Exponent.

The following are obtained by actual multiplication:

$$(a+x)^2 = a^2 + 2\,ax + x^2;$$

$$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3;$$

$$(a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4;\ \text{etc.}$$

In these results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of x in the term increased by 1, the quotient will be the coefficient of the next following term.

181. If the laws of § 180 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1 \cdot 2}$ as the coefficient of the third term; and so on.

Then,
$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots. \quad (1)$$

Multiplying both members of (1) by $a+x$, we have

$$(a+x)^{n+1} = a^{n+1} + na^n x + \frac{n(n-1)}{1 \cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots$$
$$+ a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^3 + \cdots.$$

Collecting the terms which contain like powers of a and x, we have

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + \left[\frac{n(n-1)}{1 \cdot 2} + n\right]a^{n-1}x^2$$
$$+ \left[\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \frac{n(n-1)}{1 \cdot 2}\right]a^{n-2}x^3 + \cdots$$
$$= a^{n+1} + (n+1)a^n x + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2$$
$$+ \frac{n(n-1)}{1 \cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots.$$

Then,
$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + n\left[\frac{n+1}{2}\right]a^{n-1}x^2$$
$$+ \frac{n(n-1)}{1 \cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3 + \cdots$$
$$= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1 \cdot 2}a^{n-1}x^2$$
$$+ \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots, \quad (2)$$

It will be observed that this result is in accordance with the laws of § 180; which proves that, if the laws hold for any power of $a+x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

In place of the denominators $1\cdot 2$; $1\cdot 2\cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc.

The symbol $\underline{|n}$, read "factorial n," signifies the product of the natural numbers from 1 to n, inclusive.

The method of proof exemplified in § 181 is known as *Mathematical Induction.*

182. Putting $a=1$ in equation (1), § 181, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots.$$

183. In expanding expressions by the Binomial Theorem, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 180.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second, 5.

Multiplying 5, the coefficient of the second term, by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $(a+x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$

It will be observed that the coefficients of terms equally distant from the ends of the expansion are equal; this law will be proved in § 185.

Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be enclosed, negative sign and all, in parentheses before applying the laws.

2. Expand $(1-x)^6$.

$$\begin{aligned}(1-x)^6 &= [1+(-x)]^6\\ &= 1^6+6\cdot 1^5\cdot(-x)+15\cdot 1^4\cdot(-x)^2+20\cdot 1^3\cdot(-x)^3\\ &\quad +15\cdot 1^2\cdot(-x)^4+6\cdot 1\cdot(-x)^5+(-x)^6\\ &= 1-6x+15x^2-20x^3+15x^4-6x^5+x^6.\end{aligned}$$

If the first term of the binomial is an arithmetical number, it is convenient to write the exponents at first without reduction; the result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

3. Expand $(3m^2-\sqrt[3]{n})^4$.

$$\begin{aligned}(3m^2-\sqrt[3]{n})^4 &= [(3m^2)+(-n^{\frac{1}{3}})]^4\\ &= (3m^2)^4+4(3m^2)^3(-n^{\frac{1}{3}})+6(3m^2)^2(-n^{\frac{1}{3}})^2\\ &\quad +4(3m^2)(-n^{\frac{1}{3}})^3+(-n^{\frac{1}{3}})^4\\ &= 81m^8-108m^6n^{\frac{1}{3}}+54m^4n^{\frac{2}{3}}-12m^2n+n^{\frac{4}{3}}.\end{aligned}$$

A *trinomial* may be raised to any power by the Binomial Theorem, if two of its terms be enclosed in parentheses, and regarded as a single term; but for second powers, the method of § 167 is shorter.

4. Expand $(x^2-2x-2)^4$.

$$\begin{aligned}(x^2-2x-2)^4 &= [(x^2-2x)+(-2)]^4\\ &= (x^2-2x)^4+4(x^2-2x)^3(-2)+6(x^2-2x)^2(-2)^2\\ &\quad +4(x^2-2x)(-2)^3+(-2)^4\\ &= x^8-8x^7+24x^6-32x^5+16x^4\\ &\quad -8(x^6-6x^5+12x^4-8x^3)\\ &\quad +24(x^4-4x^3+4x^2)-32(x^2-2x)+16\\ &= x^8-8x^7+16x^6+16x^5-56x^4-32x^3+64x^2+64x+16.\end{aligned}$$

EXERCISE 34

Expand the following:

1. $(a+b)^6$.
2. $(x-y)^9$.
3. $(1-x)^7$.
4. $(xy+z)^8$.
5. $(a^2-b)^5$.
6. $(2a-b)^5$.
7. $(3m-4n)^4$.
8. $(p^{\frac{1}{2}}-2q)^6$.
9. $(x^{-2}+y^{\frac{3}{2}})^5$
10. $(2a^{-\frac{1}{5}}+b^{\frac{1}{3}})^7$.
11. $\left(\frac{x^2}{2}-3y^{-1}\right)^6$.
12. $\left(\frac{a^2}{b}-\frac{b^2}{a}\right)^8$.
13. $\left(\frac{m}{3n}+3\sqrt{mn}\right)^4$.
14. $(2c^{-\frac{2}{3}}-\frac{1}{5}d^{-\frac{4}{5}})^5$.
15. $(y-z^2)^{10}$.
16. $(1-a^2)^{12}$.
17. $(\sqrt[4]{a^3}-\frac{1}{4}\sqrt[3]{a^2})^4$.
18. $(a+b)^{15}$.
19. $\left(\frac{1}{2}a^{-\frac{1}{3}}+\frac{3}{4a^{-\frac{2}{3}}}\right)^5$.
20. $(x^{\frac{5}{3}}-y^4z^3)^{11}$.
21. $(a+b-c)^4$.
22. $(x^2-2x-3)^4$.
23. $(m^2-2m+1)^4$.
24. $(x^2+x+1)^5$.
25. $(1+c+c^2)^6$.

184. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws hold for any term in the expansion of $(a+x)^n$, in equation (1), § 181:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore in the rth term, the exponent of x will be $r-1$.

The exponent of a will be $n-(r-1)$, or $n-r+1$.

The last factor of the numerator will be $n-r+2$.

The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}. \qquad (1)$$

In finding any term of an expansion, it is convenient to obtain the coefficient and exponents of the terms by the above laws.

Ex. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.
In this case, $n=11$, $r=8$.
The exponent of $(-b^{-1})$ is $8-1$, or 7.
The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.
The first factor of the numerator is 11, and the last factor $4+1$, or 5.
The last factor of the denominator is 7.

Then, the 8th term $=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$

$$=330(81a^2)(-b^{-7})=-26730a^2b^{-7}.$$

If the second term of the binomial is negative, it should be enclosed, sign and all, in parentheses before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in parentheses before applying the laws.

EXERCISE 35

Find the:

1. 5th term of $(a+b)^9$.
2. 7th term of $(x-y)^{10}$.
3. 6th term of $(1-x)^{11}$.
4. 4th term of $(a^2-b^3)^8$.
5. 8th term of $(c^{\frac{1}{2}}-2d^3)^{12}$.
6. 10th term of $\left(m^{\frac{2}{3}}+\frac{n}{2}\right)^{14}$.
7. 7th term of $\left(c^{-2}-\frac{1}{3x}\right)^{12}$.
8. 6th term of $\left(\frac{a^2}{b^2}+\frac{b}{a}\right)^8$.
9. 5th term of $\left(\sqrt{\frac{a}{3}}-\sqrt{\frac{a}{2}}\right)^9$.
10. 4th term of $(x\sqrt{y}-\frac{2}{3}y^{-\frac{3}{4}})^7$.

185. Multiplying both terms of the coefficient, in (1), § 184, by the product of the natural numbers from 1 to $n-r+1$, inclusive, the coefficient of the rth term becomes

$$\frac{n(n-1)\cdots(n-r+2)\cdot(n-r+1)\cdots 2\cdot 1}{\underline{|r-1}\times 1\cdot 2\cdots(n-r+1)}=\frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}}.$$

Since the number of terms in the expansion is $n+1$, the rth term from the end is the $(n-r+2)$th from the beginning.

Then, to find the coefficient of the rth term from the end, we put in the above formula $n-r+2$ for r.

Then, the coefficient of the rth term from the end is

$$\frac{\lfloor n}{\lfloor n-r+2-1 \; \lfloor n-(n-r+2)+1}, \text{ or } \frac{\lfloor n}{\lfloor n-r+1 \; \lfloor r-1}.$$

Hence, in the expansion of $(a+x)^n$, the coefficients of terms equidistant from the ends of the expansion are equal.

186. It was proved in § 181 that, if n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots.$$

If n is a negative integer, or a positive or negative fraction, the series in the second member is infinite (§ 178); for no one of the expressions $n-1$, $n-2$, etc., can equal zero; in this case, the series gives the value of $(a+x)^n$, provided it is convergent.

As a rigorous proof of the Binomial Theorem for Fractional and Negative Exponents is too difficult for pupils at this stage of their progress, the author has thought best to omit it; any one desiring a rigorous algebraic proof of the theorem, will find it in the author's Advanced Course in Algebra, § 575.

187. Examples.

In expanding expressions by the Binomial Theorem when the exponent is fractional or negative, the exponents and coefficients of the terms may be found by the laws of § 180, which hold for all values of the exponent.

1. Expand $(a+x)^{\frac{2}{3}}$ to five terms.

The exponent of a in the first term is $\frac{2}{3}$, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$.

Multiplying $\frac{2}{3}$, the coefficient of the second term, by $-\frac{1}{3}$, the exponent of a in that term, and dividing the product by the exponent of x increased by 1, or 2, we have $-\frac{1}{9}$ as the coefficient of the third term; and so on.

Then, $(a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 - \frac{7}{243}a^{-\frac{10}{3}}x^4 + \cdots.$

2. Expand $(1 + 2x^{-\frac{1}{2}})^{-2}$ to five terms.

Enclosing $2x^{-\frac{1}{2}}$ in parentheses, we have

$$\begin{aligned}(1 + 2x^{-\frac{1}{2}})^{-2} &= [1 + (2x^{-\frac{1}{2}})]^{-2}\\ &= 1^{-2} - 2\cdot 1^{-3}\cdot(2x^{-\frac{1}{2}}) + 3\cdot 1^{-4}\cdot(2x^{-\frac{1}{2}})^2\\ &\qquad - 4\cdot 1^{-5}\cdot(2x^{-\frac{1}{2}})^3 + 5\cdot 1^{-6}\cdot(2x^{-\frac{1}{2}})^4 - \cdots\\ &= 1 - 4x^{-\frac{1}{2}} + 12x^{-1} - 32x^{-\frac{3}{2}} + 80x^{-2} + \cdots.\end{aligned}$$

By writing the exponents of 1, in expanding $[1 + (2x^{-\frac{1}{2}})]^{-2}$, we can make use of the fifth law of § 180.

3. Expand $\dfrac{1}{\sqrt[3]{a^{-1} - 3x^{\frac{1}{3}}}}$ to four terms.

Enclosing a^{-1} and $-3x^{\frac{1}{3}}$ in parentheses, we have

$$\begin{aligned}\frac{1}{\sqrt[3]{a^{-1} - 3x^{\frac{1}{3}}}} = \frac{1}{(a^{-1} - 3x^{\frac{1}{3}})^{\frac{1}{3}}} &= [(a^{-1}) + (-3x^{\frac{1}{3}})]^{-\frac{1}{3}}\\ &= (a^{-1})^{-\frac{1}{3}} - \tfrac{1}{3}(a^{-1})^{-\frac{4}{3}}(-3x^{\frac{1}{3}}) + \tfrac{2}{9}(a^{-1})^{-\frac{7}{3}}(-3x^{\frac{1}{3}})^2\\ &\qquad - \tfrac{14}{81}(a^{-1})^{-\frac{10}{3}}(-3x^{\frac{1}{3}})^3 + \cdots\\ &= a^{\frac{1}{3}} + a^{\frac{4}{3}}x^{\frac{1}{3}} + 2a^{\frac{7}{3}}x^{\frac{2}{3}} + \tfrac{14}{3}a^{\frac{10}{3}}x + \cdots.\end{aligned}$$

EXERCISE 36

Expand each of the following to five terms:

1. $(a+x)^{\frac{3}{2}}$.

2. $(1+x)^{-8}$.

3. $(1-x)^{-\frac{3}{4}}$.

4. $\sqrt[5]{a-b}$.

5. $\dfrac{1}{(a+x)^5}$.

6. $\dfrac{1}{\sqrt[6]{1-x}}$.

7. $(a^{\frac{4}{3}} + 2b)^{\frac{3}{4}}$.

8. $(a^3 - 4x^{\frac{1}{2}})^{-\frac{7}{2}}$.

9. $\dfrac{1}{x^{-\frac{2}{3}} + 3y}$.

10. $\left(m^{-3}+\frac{n^{-\frac{5}{6}}}{4}\right)^{-3}$.

11. $\sqrt[3]{[(a^{-2}-6b^2c)^7]}$.

12. $\frac{1}{(x^{-\frac{1}{3}}-2y^{\frac{3}{2}})^4}$.

13. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^{-\frac{1}{5}}$.

14. $(m^{\frac{5}{3}}-3n^{-\frac{4}{3}})^{-\frac{4}{3}}$.

15. $\left(\frac{1}{5\sqrt[5]{a^4}}-\sqrt[3]{b^2}\right)^{\frac{3}{5}}$.

188. The formula for the rth term of $(a+x)^n$ (§ 184) holds for fractional or negative values of n, since it was derived from an expansion which holds for all values of the exponent.

Ex. Find the 7th term of $(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

Enclosing $-3x^{-\frac{3}{2}}$ in parentheses, we have

$$(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}=[a+(-3x^{-\frac{3}{2}})]^{-\frac{1}{3}}.$$

The exponent of $(-3x^{-\frac{3}{2}})$ is $7-1$, or 6.
The exponent of a is $-\frac{1}{3}-6$, or $-\frac{19}{3}$.
The first factor of the numerator is $-\frac{1}{3}$, and the last factor $-\frac{19}{3}+1$, or $-\frac{16}{3}$.
The last factor of the denominator is 6.
Hence, the 7th term

$$=\frac{\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}\cdot-\frac{10}{3}\cdot-\frac{10}{3}\cdot-\frac{16}{3}}{1\cdot2\cdot3\cdot4\cdot5\cdot6}a^{-\frac{19}{3}}(-3x^{-\frac{3}{2}})^6$$

$$=\frac{728}{3^8}a^{-\frac{19}{3}}(3^6x^{-9})=\frac{728}{9}a^{-\frac{19}{3}}x^{-9}.$$

EXERCISE 37

Find the:

1. 6th term of $(a+x)^{\frac{2}{3}}$.
2. 5th term of $(a-b)^{-\frac{1}{5}}$.
3. 7th term of $(1+x)^{-7}$.
4. 8th term of $(1-x)^{\frac{3}{5}}$.
5. 9th term of $(a-x)^{-3}$.
6. 11th term of $\sqrt{(m+n)^5}$.
7. 7th term of $(a^{-2}-2b^{\frac{1}{3}})^{-2}$.
8. 8th term of $\frac{1}{(x^5+y^{-\frac{1}{2}})^4}$.
9. 10th term of $(x^{-5}+y^{\frac{2}{3}})^{-\frac{5}{2}}$.
10. 6th term of $(a^{\frac{3}{2}}-2b^{-4})^{-\frac{3}{4}}$.
11. 5th term of $(m+3n^{-3})^{\frac{8}{3}}$.
12. 9th term of $\frac{1}{\sqrt[3]{[(a^3+3b^{-\frac{2}{3}})^5]}}$.

189. Extraction of Roots.

The Binomial Theorem may sometimes be used to find the approximate root of a number which is not a perfect power of the same degree as the index of the root.

Ex. Find $\sqrt[3]{25}$ approximately to five places of decimals.

The nearest perfect cube to 25 is 27.

We have $\sqrt[3]{25} = \sqrt[3]{27-2} = [(3^3)+(-2)]^{\frac{1}{3}}$

$$= (3^3)^{\frac{1}{3}} + \tfrac{1}{3}(3^3)^{-\frac{2}{3}}(-2) - \tfrac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2$$

$$+ \tfrac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3 - \cdots$$

$$= 3 - \frac{2}{3\cdot 3^2} - \frac{4}{9\cdot 3^5} - \frac{40}{81\cdot 3^8} - \cdots.$$

Expressing each fraction approximately to the nearest fifth decimal place, we have

$$\sqrt[3]{25} = 3 - .07407 - .00183 - .00008 - \cdots = 2.92402.$$

We then have the following rule:

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the Binomial Theorem.

If the ratio of the second term of the binomial to the first is a small proper fraction, the terms of the expansion diminish rapidly; but if this ratio is but little less than 1, it requires a great many terms to insure any degree of accuracy.

EXERCISE 38

Find the approximate values of the following to five places of decimals:

1. $\sqrt{17}$. 2. $\sqrt{51}$. 3. $\sqrt[3]{60}$. 4. $\sqrt[4]{14}$. 5. $\sqrt[4]{84}$. 6. $\sqrt[5]{35}$.

PROPERTIES OF QUADRATIC SURDS

190. A quadratic surd (§ 70) cannot equal the sum of a rational expression and a quadratic surd.

For, if possible, let $(a)^{\frac{1}{2}} = b + (c)^{\frac{1}{2}}$,

where b is a rational expression, and $(a)^{\frac{1}{2}}$ and $(c)^{\frac{1}{2}}$ quadratic surds.

Squaring both members, $a = b^2 + 2b(c)^{\frac{1}{2}} + c,$

or, $2b(c)^{\frac{1}{2}} = a - b^2 - c.$

Whence, $(c)^{\frac{1}{2}} = \dfrac{a - b^2 - c}{2b}.$

That is, a quadratic surd equal to a rational expression.

But this is impossible; whence, $(a)^{\frac{1}{2}}$ cannot equal $b + (c)^{\frac{1}{2}}$.

191. If $a + (b)^{\frac{1}{2}} = c + (d)^{\frac{1}{2}}$, where a and c are rational expressions, and $(b)^{\frac{1}{2}}$ and $(d)^{\frac{1}{2}}$ quadratic surds, then

$$a = c, \text{ and } (b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}.$$

If a does not equal c, let $a = c + x$; then, x is rational.

Substituting this value in the given equation,

$$c + x + (b)^{\frac{1}{2}} = c + (d)^{\frac{1}{2}}, \text{ or } x + (b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}.$$

But this is impossible by § 190.

Then, $a = c$, and therefore $(b)^{\frac{1}{2}} = (d)^{\frac{1}{2}}$.

192. If $(a + \sqrt{b})^{\frac{1}{2}} = \sqrt{x} + \sqrt{y}$, where a, b, x, and y are rational expressions, then $(a - \sqrt{b})^{\frac{1}{2}} = \sqrt{x} - \sqrt{y}$.

Squaring both members of the given equation,

$$a + \sqrt{b} = x + 2\sqrt{xy} + y,$$

Whence, by § 191, $a = x + y,$

and $(b)^{\frac{1}{2}} = 2(xy)^{\frac{1}{2}}.$

Subtracting, $a - (b)^{\frac{1}{2}} = x - 2(xy)^{\frac{1}{2}} + y.$

Extracting the square root of both members,

$$(a - \sqrt{b})^{\frac{1}{2}} = \sqrt{x} - \sqrt{y}.$$

193. Square Root of a Binomial Surd.

The preceding principles may be used to find the square roots of certain expressions which are in the form of the sum of a rational expression and a quadratic surd.

Ex. Find the square root of $13-\sqrt{160}$.

Assume, $\sqrt{13-\sqrt{160}}=\sqrt{x}-\sqrt{y}$. (1)

Then, by § 192, $\sqrt{13+\sqrt{160}}=\sqrt{x}+\sqrt{y}$. (2)

Multiply (1) by (2), $\sqrt{169-160}=x-y$.

Or, $x-y=3$. (3)

Squaring (1) $13-\sqrt{160}=x-2\sqrt{xy}+y$.

Whence, by § 191, $x+y=13$. (4)

Adding (3) and (4), $2x=16$, or $x=8$.

Subtracting (3) from (4), $2y=10$, or $y=5$.

Substitute in (1), $\sqrt{13-\sqrt{160}}=\sqrt{8}-\sqrt{5}=2\sqrt{2}-\sqrt{5}$.

194. Examples like that of § 193 may be solved by inspection, by putting the given expression into the form of a trinomial perfect square (§ 103, II), as follows:

Reduce the surd term so that its coefficient may be 2.

Separate the rational term into two parts whose product shall be the expression under the radical sign of the surd term.

Extract the square root of each part, and connect the results by the sign of the surd term.

1. Extract the square root of $8+\sqrt{48}$.

We have $\sqrt{48}=2\sqrt{12}$.

We then separate 8 into two parts whose product is 12.

The parts are 6 and 2; whence,

$$\sqrt{8+\sqrt{48}}=\sqrt{6+2\sqrt{12}+2}=\sqrt{6}+\sqrt{2}.$$

2. Extract the square root of $22-3\sqrt{32}$.

We have $3\sqrt{32}=\sqrt{9\times 8\times 4}=2\sqrt{72}$.

We then separate 22 into two parts whose product is 72.

The parts are 18 and 4; whence,

$$\sqrt{22-3\sqrt{32}}=\sqrt{18-2\sqrt{72}+4}=\sqrt{18}-\sqrt{4}=3\sqrt{2}-2.$$

EXERCISE 39

Find the square roots of the following:

1. $5+2\sqrt{6}$.
2. $8-2\sqrt{12}$.
3. $8a-2a\sqrt{15}$.
4. $9+2(14)^{\frac{1}{2}}$.
5. $7+4(3)^{\frac{1}{2}}$.
6. $17-12\sqrt{2}$.
7. $2+(3)^{\frac{1}{2}}$.
8. $1+\frac{1}{2}\sqrt{3}$.
9. $2c-2(c^2-d^2)^{\frac{1}{2}}$.
10. $m+2\sqrt{mn-n^2}$.
11. $a-\sqrt{a^2-b^2}$.
12. $5x+x\sqrt{21}$.
13. $113-12(85)^{\frac{1}{2}}$.
14. $366+24\sqrt{210}$.
15. $540-14\sqrt{11}$.

195. Solution of Equations having the Unknown Numbers under Radical Signs.

1. Solve the equation $\sqrt{x^2-5}-x=-1$.

Transposing $-x$, $\quad \sqrt{x^2-5}=x-1$.

Squaring both members, $\quad x^2-5=x^2-2x+1$.

Transposing, $\quad 2x=6$; whence, $x=3$.

(Substituting 3 for x in the given first member, and taking the positive value of the square root, the first member becomes

$$\sqrt{9-5}-3=2-3=-1;$$

which shows that the solution $x=3$ is correct.)

We then have the following rule:

Transpose the terms of the equation so that a surd term may stand alone in one member; then raise both members to a power of the same degree as the surd.

If surd terms still remain, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$, $\sqrt{2x+6}=7-\sqrt{2x-1}$.

Squaring, $2x+6=49-14\sqrt{2x-1}+2x-1$.

Transposing, $14\sqrt{2x-1}=42$, or $\sqrt{2x-1}=3$.

Squaring, $2x-1=9$; whence, $x=5$.

3. Solve the equation $\sqrt{x-2}-\sqrt{x}=\dfrac{1}{\sqrt{x-2}}$.

Clearing of fractions, $x-2-\sqrt{x^2-2x}=1$.

Transposing, $-\sqrt{x^2-2x}=3-x$.

Squaring, $x^2-2x=9-6x+x^2$.

Transposing, $4x=9$, and $x=\frac{9}{4}$.

(If we put $x=\frac{9}{4}$, the given equation becomes

$$\sqrt{\tfrac{1}{4}}-\sqrt{\tfrac{9}{4}}=\frac{1}{\sqrt{\tfrac{1}{4}}}.\Big) \qquad (1)$$

If we take the *positive* value of each square root, the above is not a true equation.

Authorities differ as to whether it is allowable in such instance to choose the *negative* value for one of the square roots. It seems more consistent to adhere to the signs expressed in the given equation. If this rule is followed, the above equation has no solution.

EXERCISE 40

Solve the following equations; verify each root:

1. $\sqrt{x+5}+2=5$.
2. $\sqrt{x+7}-\sqrt{x}=1$.
3. $\sqrt{x^2+4x-3}-\sqrt{x^2+x+6}=0$.
4. $\sqrt[3]{x^2+11}+4=7$.
5. $\sqrt{x^2-11}+1=x$.
6. $\sqrt{x-28}=14-\sqrt{x}$.
7. $\sqrt{x}+\sqrt{10-x}=\dfrac{12}{\sqrt{10-x}}$.
8. $\sqrt{x}+\sqrt{x+8}=\dfrac{12}{\sqrt{x+8}}$.
9. $\dfrac{\sqrt{x}+11}{\sqrt{x}-3}=\dfrac{\sqrt{x}+19}{\sqrt{x}-2}$.
10. $\dfrac{\sqrt{4x+5}+\sqrt{x+3}}{\sqrt{4x+5}-\sqrt{x+3}}=5$.

11. $\dfrac{3\sqrt{2x}+4}{\sqrt{2x}}=\dfrac{3\sqrt{2x}+2}{\sqrt{2x}+1}.$ 12. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{1}{2}.$

13. $\sqrt{x+m}+\sqrt{x-n}=\sqrt{2m-n+3x}.$

14. $\sqrt{a-y}+\sqrt{b-y}=\sqrt{a-b}.$

15. $\sqrt{2s+3}-\sqrt{3s+3}=-\sqrt{s-10}.$

16. $\sqrt{1+x\sqrt{4-x}}=1+x.$

17. $x\sqrt{x-1-\sqrt{x}}=x.$

18. $\dfrac{\sqrt{b+x}+\sqrt{x}}{\sqrt{b+x}-\sqrt{x}}=b.$

19. $\sqrt[4]{x^2-5x-8}=\sqrt{x-4}.$

20. $\dfrac{\sqrt{b^2+x}+\sqrt{c^2+x}}{\sqrt{b^2+x}-\sqrt{c^2+x}}=\dfrac{b}{c}.$

VII. IMAGINARY NUMBERS

196. If a number involves an indicated even root of a negative number, it is called **imaginary.** Such numbers depend upon a new unit, $\sqrt{-1}$ or $(-1)^{\frac{1}{2}}$; as $\sqrt{-2}$, $\sqrt[4]{-3}$.

197. An imaginary number of the form $\sqrt{-a}$ is called a **pure imaginary** number, and the sum of a real and an imaginary is called a **complex** number; as $a+b\sqrt{-1}$.

198. Meaning of a Pure Imaginary Number.

If $\sqrt{a}$ is *real,* we define $\sqrt{a}$ as an expression such that, when raised to the second power, the result is a.

To find what meaning to attach to a pure imaginary number, we assume the above principle to hold when $\sqrt{a}$ is imaginary.

Thus, $\sqrt{-2}$ means an expression such that, when raised to the second power, the result is -2; that is, $(\sqrt{-2})^2$ or $(-2^{\frac{1}{2}})^2=-2$.

In like manner, $(\sqrt{-1})^2=(-1^{\frac{1}{2}})^2=-1$; etc.

OPERATIONS WITH IMAGINARY NUMBERS

199. By § 198, $(\sqrt{-5})^2 = (-5^{\frac{1}{2}})^2 = -5.$ (1)

Also, $(\sqrt{5}\sqrt{-1})^2 = (\sqrt{5})^2(\sqrt{-1})^2 = 5(-1) = -5,$ (2)

or $(\sqrt{-5})^2 = (5^{\frac{1}{2}})^2 \cdot (-1^{\frac{1}{2}})^2 = 5(-1) = -5.$

From (1) and (2), $(\sqrt{-5})^2 = (\sqrt{5}\sqrt{-1})^2.$

Whence, $\sqrt{-5} = \sqrt{5}\sqrt{-1}$, or $5^{\frac{1}{2}}(-1)^{\frac{1}{2}}.$

Then, *every imaginary square root can be expressed as the product of a real number by* $\sqrt{-1}$. *It is advisable to reduce every imaginary to this form before performing the indicated operations.*

$\sqrt{-1}$ is called the *imaginary unit;* it is often represented by i.

In all operations with imaginary numbers, it is advisable to reduce the number to the form $a + bi$ where a and b are real.

Ex. Add $\sqrt{-4}$ and $\sqrt{-36}$.

$$\sqrt{-4} = 2\,i,\ \sqrt{-36} = 6\,i.$$

$$2\,i + 6\,i = 8\,i, \text{ or } 8\sqrt{-1}, \text{ or } 8\,(-1)^{\frac{1}{2}}.$$

The Powers of i:

$$\sqrt{-1} = i^1;$$

$$(\sqrt{-1})^2 = -1 = i^2;$$

$$(\sqrt{-1})^3 = -\sqrt{-1} = i^3;$$

$$(\sqrt{-1})^4 = 1 = i^4;$$

$$(\sqrt{-1})^5 = \sqrt{-1} = i^5.$$

Note that the even powers of i are real, the odd powers imaginary, the fifth power like the first power, the sixth power like the second, etc.

Ex. 1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

$$\sqrt{-2} = i\,\sqrt{2},\ \ \sqrt{-3} = i\,\sqrt{3}.$$

$$(i\sqrt{2})\,(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}, \text{ or } -(6)^{\frac{1}{2}}.$$

Ex. 2. Divide $(-40)^{\frac{1}{2}}$ by $(-5)^{\frac{1}{2}}$.

$$(-40)^{\frac{1}{2}} = i\,(40)^{\frac{1}{2}},\ (-5)^{\frac{1}{2}} = i\,(5)^{\frac{1}{2}}$$

$$\frac{i\,(40)^{\frac{1}{2}}}{i\,(5)^{\frac{1}{2}}} = (8)^{\frac{1}{2}} = 2\,(2)^{\frac{1}{2}},\text{ or } 2\sqrt{2}.$$

EXERCISE 41

Simplify the following:

1. $\sqrt{-16} + \sqrt{-4}$.
2. $2\sqrt{-9} + 4\sqrt{-25} - 3\sqrt{-36}$.
3. $2\sqrt{-3} - 3\sqrt{-27} + 5\sqrt{-12}$.
4. $7\sqrt{-a^2} - 3\sqrt{-49\,a^2} - 2\sqrt{-4\,a^2}$.
5. $\frac{1}{3}\sqrt{-8} + \frac{1}{5}\sqrt{-32} - \frac{1}{6}\sqrt{-162}$.
6. Add $2 + \sqrt{-12}$ to $3 - 2\sqrt{-27}$.
7. From $8 - 6\sqrt{-121}$ subtract $5 + 2\sqrt{-169}$.
8. From $a - \sqrt{2\,b - b^2 - 1}$ take $b - \sqrt{2\,a - a^2 - 1}$.

Multiply the following:

9. $\sqrt{-2}$ by $\sqrt{-7}$.
10. $\sqrt{-4}$ by $\sqrt{-144}$.
11. $-\sqrt{-27}$ by $\sqrt{-6}$.
12. $-\sqrt{-432}$ by $-\sqrt{-75}$.
13. $\sqrt{-a^2}$, $\sqrt{-b^2}$, and $-\sqrt{-c^2}$.
14. $2 + \sqrt{-3}$ by $3 - 4\sqrt{-3}$.
15. $5 - 2\sqrt{-4}$ by $4 - 3\,i$.
16. $4\,i\sqrt{x} - 3\,i\sqrt{y}$ by $9\,i\sqrt{x} + \sqrt{-y}$.

Expand the following:

17. $(2 - \sqrt{-3})^2$.
18. $(3\sqrt{-2} + 2\sqrt{-3})^2$.
19. $(2\sqrt{-5} + 3\sqrt{-7})(2\sqrt{-5} - 3\sqrt{-7})$.
20. $(a - \sqrt{-b})^3$.

Divide the following:

21. $\sqrt{-18}$ by $\sqrt{-2}$.

22. $\sqrt{-54}$ by $-\sqrt{-3}$.

23. $-\sqrt{192}$ by $-\sqrt{-3}$.

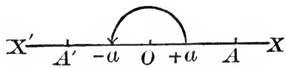
24. $-\sqrt{-96\,ab}$ by $\sqrt{-3\,ab}$.

25. $6\,i\sqrt{6}-\sqrt{384}$ by $-\sqrt{-6}$.

GRAPHICAL REPRESENTATION OF IMAGINARY NUMBERS

200. Let O be any point in the straight line XX'.

We may suppose any positive real number, $+a$, to be represented by the distance from O to A, a units to the right of O in OX.

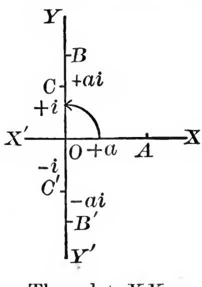


Then any negative real number, $-a$, may be represented by the distance from O to A', a units to the left of O in OX'.

201. Since $-a$ is the same as $(+a)\times(-1)$, it follows from § 200 that the product of $+a$ by -1 is represented by turning the line OA, which represents the number $+a$, through two right angles, in a direction opposite to the motion of the hands of a clock.

Then, in the product of any real number by -1, we may regard -1 as an operator which turns the line which represents the first factor through two right angles, in a direction opposite to the motion of the hands of a clock.

202. Graphical Representation of the Imaginary Unit i (§ 196).



By the definition of § 198, $-1 = i \times i$.

Then, since one multiplication by i, followed by another multiplication by i, turns the line which represents the first factor through *two* right angles, in a direction opposite to the hands of a clock, we may regard multiplication by i as turning the line through *one* right angle, in the same direction.

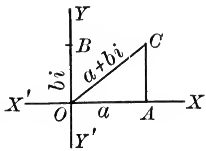
Thus, let XX' and YY' be straight lines intersecting at right angles at O.

Then, if $+a$ be represented by the line OA, where A is a units to the right of O in OX, $+ai$ may be represented by OB, and $-ai$ by OB', where B is a units above, and B' a units below, O, in YY'.

Also, $+i$ may be represented by OC, and $-i$ by OC', where C is one unit above, and C' one unit below, O, in YY'.

203. Graphical Representation of Complex Numbers.

We will now show how to represent the complex number $a+bi$.



Let XX' and YY' be straight lines intersecting at right angles at O.

Let a be represented by OA, to the right of O, if a is positive, to the left if a is negative.

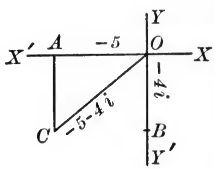
Let bi be represented by OB, above O if b is positive, below if b is negative.

Draw line AC equal and parallel to OB, on the same side of XX' as OB, and line OC.

Then, OC is considered as representing the result of adding bi to a; that is, OC represents the complex number $a+bi$.

The figure represents the case in which both a and b are *positive.*

As another illustration, we will show how to represent the complex number $-5-4i$.



Lay off OA 5 units to the left of O in OX', and OB 4 units below O in YY'.

Draw line AC below XX', equal and parallel to OB, and line OC.

Then, OC represents $-5-4i$.

The complex number $a+bi$, if a is positive and b negative, will be represented by a line between OX and OY'; and if a is negative and b positive, by a line between OY and OX'.

EXERCISE 42

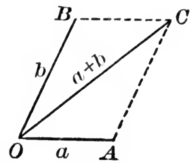
Represent the following graphically:

1. $3i$. 2. $-6i$. 3. $4+i$. 4. $-1+2i$.

5. $2-5i$. 6. $-5-3i$. 7. $-7+4i$.

204. Graphical Representation of Addition.

We will now show how to represent the result of adding b to a, where a and b are any two real, pure imaginary, or complex numbers.



Let the line a be represented by OA, and the line b by OB.

Draw the line AC equal and parallel to OB, on the same side of OA as OB, and the line OC.

Then, OC is considered as representing the result of adding b to a; that is, OC represents $a + b$.

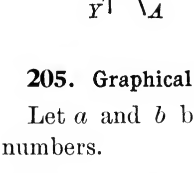
The method of § 203 is a special case of the above.

If a and b are both real, B will fall in OA, or in AO produced through O.

The same will be true if a and b are both pure imaginary.

If one of the numbers, a and b, is real, and the other pure imaginary, the lines OA and OB will be perpendicular.

As another illustration, we will show how to represent graphically the sum of the complex numbers $2 - 5i$ and $-4 + 3i$.



The complex number $2 - 5i$ is represented by the line OA, between OX and OY'.

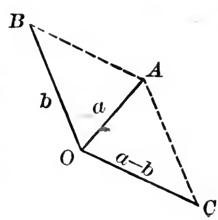
The complex number $-4 + 3i$ is represented by the line OB, between OY and OX'.

Draw the line BC equal and parallel to OA, on the same side of OB as OA, and the line OC.

Then, the line OC represents the result of adding $-4 + 3i$ to $2 - 5i$.

205. Graphical Representation of Subtraction.

Let a and b be any two real, pure imaginary, or complex numbers.



Let a be represented by OA, and b by OB; and complete the parallelogram $OBAC$.

By § 204, OA represents the result of adding the number represented by OB to the number represented by OC.

That is, if b be added to the number represented by OC, the sum is equal to a; hence, $a - b$ is represented by the line OC.

EXERCISE 43

Represent the following graphically:

1. The sum of $4i$ and $3-5i$.
2. The sum of $-5i$ and $-1+6i$.
3. The sum of $2+4i$ and $5-3i$.
4. The sum of $-6+2i$ and $-4-7i$.

5. Represent graphically the result of *subtracting* the second expression from the first, in each of the above examples.

VIII. QUADRATIC EQUATIONS

206. A **quadratic equation** is an equation in which the highest power of the unknown number is the second.

207. The first power of the unknown number may or may not appear. If the equation does not contain the first degree of the unknown, the roots are of the same absolute value but of different sign. *E.g.* $x^2=a^2$; then, $(x+a)(x-a)=0$, or $x=a$, $x=-a$.

The equation may also be solved by extracting the square root of each member of the equation, whence, $x=\pm a$.

208. If the equation contains both the first and second powers of the unknown, the first member must be reduced to the form $a^2+2ab+b^2$ before extracting the square root. Such transformation of the equation is called completing the square.

209. A quadratic equation containing the first and second powers of the unknown number is called an **affected quadratic.** An equation containing the second degree only of the unknown number is a **pure quadratic.**

AFFECTED QUADRATIC EQUATIONS

210. First Method of Completing the Square.

By transposing the terms involving x to the first member, and all other terms to the second, and then dividing both members by the coefficient of x^2, any affected quadratic equation can be reduced to the form $x^2 + px = q$.

We then add to both members such an expression as will make the first member a trinomial perfect square (§ 103, II); an operation which is termed *completing the square.*

Ex. Solve the equation $x^2 + 3x = 4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots (§ 103, II).

Then, the square root of the third term is equal to the second term divided by twice the square root of the first.

Hence, the *square root* of the expression which must be added to $x^2 + 3x$ to make it a perfect square is $3x \div 2x$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2 + 3x + (\tfrac{3}{2})^2 = 4 + \tfrac{9}{4} = \tfrac{25}{4}.$$

Equating the square root of the first member to the $\pm$ square root of the second, we have

$$x + \tfrac{3}{2} = \pm \tfrac{5}{2}.$$

Transposing $\frac{3}{2}$, $\quad x = -\frac{3}{2} + \frac{5}{2}$ or $-\frac{3}{2} - \frac{5}{2} = 1$ or -4.

We then have the following rule:

Reduce the equation to the form $x^2 + px = q$.

Complete the square, by adding to both members the square of one-half the coefficient of x.

Equate the square root of the first member to the $\pm$ square root of the second, and solve the linear equations thus formed.

211. The objection to the method of § 210 is that in dividing by the coefficient of x^2, or in adding the square of one-half the coefficient of x, fractions which make the solution cumbersome may be introduced.

212. If the coefficient of x^2 is a perfect square, it is sometimes convenient to complete the square directly by the principle stated in § 210; that is, *by adding to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2.*

Ex. Solve the equation $9x^2 - 5x = 4$.

Adding to both members the square of $\frac{5}{2 \times 3}$, or $\frac{5}{6}$,

$$9x^2 - 5x + (\tfrac{5}{6})^2 = 4 + \tfrac{25}{36} = \tfrac{169}{36}.$$

Extracting square roots, $3x - \frac{5}{6} = \pm \frac{13}{6}$.

Then, $3x = \frac{5}{6} \pm \frac{13}{6} = 3$ or $\frac{4}{3}$, and $x = 1$ or $-\frac{4}{9}$.

213. Second Method of Completing the Square.

Every affected quadratic equation can be reduced to the form $ax^2 + bx + c = 0$, or $ax^2 + bx = -c$.

Multiplying both members by $4a$, we have

$$4a^2x^2 + 4abx = -4ac.$$

We complete the square by adding to both members the square of

$$\frac{4ab}{2 \times 2a} \text{ (§ 212), or } b.$$

Then, $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$.

Extracting square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}$.

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}$.

Whence,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1)$$

We then have the following rule:

Reduce the equation to the form $ax^2 + bx = -c$. Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

The advantage of this method over the preceding is in avoiding fractions in completing the square.

This method is sometimes called the Hindoo Method.

The result of the solution of $ax^2 + bx + c = 0$ may be used as a formula for solving any quadratic equation. Before applying the formula the equation must be reduced to the form

$$ax^2 + bx + c = 0.$$

Ex. Solve $2x^2 - 7x = -3$.

$$2x^2 - 7x + 3 = 0.$$

Here $a = 2$, $b = -7$, $c = 3$; substituting in (1),

$$x = \frac{7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

EXERCISE 44

Solve by the first method: (Verify each result.)

1. $x^2 - 12x + 32 = 0$.
2. $z^2 + 7z - 30 = 0$.
3. $4y^2 - 7y = -3$.
4. $16x^2 - 8x - 35 = 0$.
5. $3m^2 - 26 = 9m^2 - 80$.
6. $t^2 + t - 30 = 0$.
7. $6z^2 + 4 = -11z$.
8. $4x^2 - 3x = 7$.
9. $\frac{z^2}{3} - \frac{z}{2} - \frac{35}{6} = 0$.
10. $\frac{3x^2}{x^2 - 7x + 6} + \frac{2x - 6}{x - 6} = 1 - \frac{2}{x - 1}$.

Solve by second method: (Verify each result.)

11. $(3k + 2)(2k + 3) = (k - 3)(2k - 4)$.
12. $\frac{30}{x} - \frac{30}{x + 1} = 1$.
13. $\sqrt{m + 2} + \sqrt{3m + 4} = 8$.
14. $(y - 3)^3 - (y + 2)^3 = -65$.
15. $\sqrt{5 + x} + \sqrt{5 - x} = \frac{12}{\sqrt{5 - x}}$.
16. $\frac{d + 3}{d - 2} - \frac{d + 4}{d} = \frac{3}{2}$.

Note 1: In solving equations involving fractions or radicals reject any root which does not satisfy the *given* equation.

Solve by means of the formula in § 213, (1): (Verify each result.)

17. $3x^2 - 2x = 40$.

18. $9x^2 + 18x = -8$.

19. $\frac{5}{6z} - \frac{13}{9z^2} = \frac{1}{18}$.

20. $\frac{1}{x+3} - \frac{1}{x-5} = \frac{x^2 - 17}{x^2 - 2x - 15}$.

21. $\frac{y-c}{y+c} - \frac{y+c}{y-c} = \frac{y^2 - 5c^2}{y^2 - c^2}$.

22. $\frac{1}{z-2} + \frac{7z}{24(z+2)} = \frac{15}{z^2 - 4}$.

23. $\frac{1}{x+a} + \frac{1}{a} + \frac{1}{x+b} + \frac{1}{b} = 0$.

24. $S = Vt + \frac{1}{2}gt^2$. Solve for t.

25. $\frac{x-2}{x-4} - \frac{x+2}{x+3} - \frac{x-2}{x-6} = -1$. (See Note 2.)

26. $\frac{x+1}{x-1} + \frac{x+2}{x-2} + \frac{x+3}{x-3} = 3$.

Note 2: In solving fractional equations containing improper fractions the operations are greatly simplified by reducing the fractions to mixed numbers and then combining the integers thus obtained.

Note 3: An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown number, and *the exponent of the unknown number in one of these terms is twice its exponent in the other*; as,

$$x^6 - 6x^3 = 16;\ x^3 + x^{\frac{3}{2}} - 72 = 0;\ \text{etc.}$$

Equations in the quadratic form may be readily solved by the rules for quadratics.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square by the rule of § 210,

$$x^6 - 6x^3 + 9 = 16 + 9 = 25.$$

Extracting square roots, $x^3 - 3 = \pm 5.$

Then, $x^3 = 3 \pm 5 = 8 \text{ or } -2.$

Extracting cube roots, $x = 2 \text{ or } -\sqrt[3]{2}.$

There are also four imaginary roots, which may be found by the method of §§ 110; 213, (1).

Solve the equation $2x + 3\sqrt{x} = 27.$

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.

Multiplying by 8, and adding 3^2 to both members § (213),

$$16x + 24\sqrt{x} + 9 = 216 + 9 = 225.$$

Extracting square roots, $4\sqrt{x} + 3 = \pm 15.$

Then, $4\sqrt{x} = -3 \pm 15 = 12 \text{ or } -18.$

Whence, $\sqrt{x} = 3 \text{ or } -\frac{9}{2}$, and $x = 9 \text{ or } \frac{81}{4}.$

EXERCISE 45

Solve the following equations and verify each root:

1. $3x^2 - 4x = 4.$

2. $7x^2 - 17x = 2x^2 + 22.$

3. $4y^2 + 9y - 13 = 0.$

4. $2t = 10 - t^2 + 5t.$

5. $6v^2 - 14v = 9v - 22.$

6. $\dfrac{x-3}{x+3} + \dfrac{x+5}{x-11} = -\dfrac{8}{7}.$

7. $\dfrac{3m-7}{m(m+2)} - \dfrac{1}{3(m-2)} = \dfrac{2-m}{m^2-4}.$

8. $\dfrac{4}{9-q} + \dfrac{15}{10-q} = 4.$

9. $\dfrac{x+1}{x+2} + \dfrac{x+2}{x+3} = 3\frac{1}{2}.$

10. $1 + 2\sqrt{3x+2} = \dfrac{21}{\sqrt{3x+2}}.$

11. $\sqrt{x-b} + 2\sqrt{2b} = \dfrac{6b}{\sqrt{x-b}}.$

12. $\dfrac{11}{3v-4a} - \dfrac{20}{2v+a} - \dfrac{3a}{6v^2-5av-4a^2} = \dfrac{6}{5a}.$

13. $\frac{a}{z-a}+\frac{b}{b-z}=\frac{a^2-b^2}{ab}$.

14. $\frac{x+a+c}{2}+\frac{3ac+6bc-9c^2}{x}=a+b+2c$.

15. $x^2-2ax+6b^2=3a^2+7ab-5bx$.

16. $(2a-b+5c)x^2+(b-a+4c)x+2b-3a-c=0$.

17. $\frac{1}{d+a}+\frac{1}{a}+\frac{1}{d+b}+\frac{1}{b}=0$. Solve for d.

18. $\frac{\sqrt{m^2+x^2}+\sqrt{m^2-x^2}}{\sqrt{m^2+x^2}-\sqrt{m^2-x^2}}=\frac{m}{x}$.

19. $x^4-7x^2+12=0$.

20. $x^6-7x^3=8$.

21. $6x^{-2}-11x^{-1}=35$.

22. $x^{\frac{1}{2}}-x^{\frac{1}{4}}-6=0$.

23. $x^3-35x^{\frac{3}{2}}=-216$.

24. $x^2+2x+10+\sqrt{x^2+2x+10}=30$.

25. $x^2+3ax-53a^2=2ax+3a$.

26. $x^{-\frac{4}{3}}-29x^{-\frac{2}{3}}=-100$.

27. $x^2+14\sqrt{x^2+7x-26}=58-7x$.

28. $5(x+2)^{\frac{1}{2}}+(x+2)=36$.

29. $(x^2+4x+2)^2=31+2(x^2+4x+2)$.

30. $x^4-8x^3+10x^2+24x-315=0$.

31. What number is that to which if you add its square the sum will be 42?

32. A rectangular field is 40 rods longer than it is wide. By doubling its length and decreasing its width by 15 rods, the area is unchanged. Find dimensions of the field.

33. The difference between two numbers is 7, and the difference between their cubes is 1267. Find the numbers.

34. The denominator of a fraction is 3 more than its numerator and by adding the fraction to its reciprocal the sum is $2\frac{9}{28}$. What is the fraction?

35. There is a number consisting of two digits whose sum is 11. If from the number 3 times the product of the digits is subtracted, the remainder will equal the sum of the digits. Find the number.

36. A man sells goods for $120, gaining a per cent equal to $\frac{1}{5}$ the cost of the goods. What was the cost of the goods?

37. A picture 13 inches by 8 inches is surrounded by a frame of uniform width whose area is 162 square inches. Find the width of the frame.

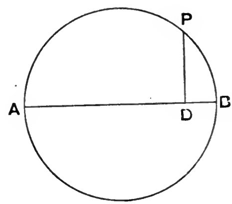
38. A man put $2400 in a savings bank which paid interest semiannually. At the end of a year he found that he had to his credit $2496.96. What interest did the bank pay?

39. A number of people plan an excursion which is to cost them $30. It is found later that 3 of the party cannot go, which increases the cost 50 cents to each member. How many are there in the party and what did each one pay?

40. A and B start together for a 6-mile walk. A's rate per hour is $\frac{1}{2}$ mile more than B's, and he finds he can reach his destination in 24 minutes less time than B. What is the rate of each?

41. An open rectangular box is 8 inches high. Its length is 4 more than its width. Its volume is 768 cubic inches. Find its inside dimensions.

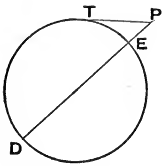
42. In a given circle APB, a perpendicular DP, dropped from a point P in the circumference to the diameter AB, is a mean proportional between the segments, AD and DB, of the diameter. If the radius of the circle is 12 and DP is $2\sqrt{5}$, how far is D from B?



43. An open rectangular box 5 inches deep (inside measure) is made of 1-inch lumber. Its length is 1 inch less than twice its width. The difference between the volumes when inside and outside measurements are taken is 271 cubic inches. How much sheet metal will be needed for lining the sides and bottom of the box?

44. Two lines AB and CD intersect at O in such a manner that $AO \cdot OB = CO \cdot OD$. If $CD = 14$, $AO = 15$, and $AB = 18$, find CO.

45. A has a lease on a square room. He sublet to B a part 10 feet wide along one entire side of the room, at a rental of $160 per month. The part of the room retained by A contained 704 square feet. How much rental per square foot did B pay? Explain your negative roots.



46. A tangent, PT, to a circle is a mean proportional between the whole secant PD and the external segment PE. If PT is 12, the radius 5, and PD passes through the center, find PE.

47. The upper base and the altitude of a trapezoid are equal, the lower base is 20 and the area is 112. Find the upper base.

48. The length of a rectangle is $\sqrt{2}$ more than the side of a given square, and its breadth is $\sqrt{2}$ less than a side of the same square. The area of the rectangle is 1. Find the dimensions of the rectangle correct to three decimal places.

THEORY OF QUADRATIC EQUATIONS

214. Number of Roots.

A quadratic equation cannot have more than two different roots.

Every quadratic equation can be reduced to the form

$$ax^2 + bx + c = 0.$$

If possible, let this have three different roots, r_1, r_2, and r_3.

Then, $$ar_1^2 + br_1 + c = 0, \quad (1)$$

$$ar_2^2 + br_2 + c = 0, \quad (2)$$

and $$ar_3^2 + br_3 + c = 0. \quad (3)$$

Subtracting (2) from (1), $a(r_1^2 - r_2^2) + b(r_1 - r_2) = 0.$

Then, $$a(r_1 + r_2)(r_1 - r_2) + b(r_1 - r_2) = 0,$$

or, $$(r_1 - r_2)(ar_1 + ar_2 + b) = 0.$$

Then, by § 110, either $r_1 - r_2 = 0$, or $ar_1 + ar_2 + b = 0$.

But $r_1 - r_2$ cannot equal 0, for, by hypothesis, r_1 and r_2 are different.

Whence, $$ar_1 + ar_2 + b = 0. \quad (4)$$

In like manner, by subtracting (3) from (1), we have

$$ar_1 + ar_3 + b = 0. \quad (5)$$

Subtracting (5) from (4), $ar_2 - ar_3 = 0$, or $r_2 - r_3 = 0$.

But this is impossible, for, by hypothesis, r_2 and r_3 are different; hence, a quadratic equation cannot have more than two different roots.

215. The graphs of quadratic equations can be readily constructed by the method used in §§ 44–48.

Construct the graph of $$x^2 - x - 6 = 0. \quad (1)$$

Placing the first member of the equation equal to y, we have

$$x^2 - x - 6 = y. \quad (2)$$

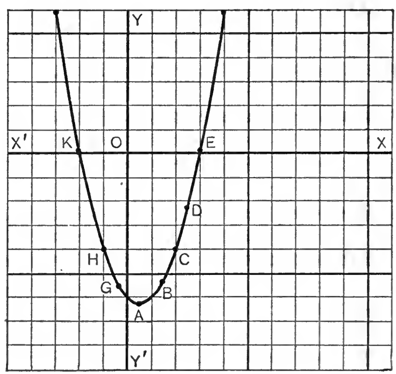
Assigning values to x, we obtain corresponding values of y. For example,

Substituting $x = 0$ in (2), we have $y = -6$,

Substituting $x = 2$ in (2), we have $y = -4$, etc.

$x^2 - x - 6 = y$

x	y
0	-6
$\frac{1}{2}$	$-6\frac{1}{4}$ (A)
1	-6
$\frac{3}{2}$	$-5\frac{1}{4}$ (B)
2	-4 (C)
$\frac{5}{2}$	$-2\frac{1}{4}$ (D)
3	0 (E)
4	6
$-\frac{1}{2}$	$-5\frac{1}{4}$ (G)
-1	-4 (H)
-2	0 (K)
-3	6



Solving $$x^2 - x - 6 = 0,$$

or $$(x - 3)(x + 2) = 0,$$

we have, $$x = 3 \text{ or } -2.$$

These values, $x = 3$, $x = -2$, are the abscissas of the points where the curve crosses the x-axis, the curve showing in a graphical way why a quadratic equation has two roots.

The graph of every equation of the form $x^2 + px - q = 0$ or $ax^2 + bx + c = 0$ is a curve of the above form and is called a *parabola*.

216. Sum of Roots and Product of Roots.

Let r_1 and r_2 denote the roots of $ax^2 + bx + c = 0$.

By § 213, (1), $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Adding these values, $$r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}.$$

Multiplying them together,

$$r_1 r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} \text{ (§ 103, I)} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, **if a quadratic equation is in the form $ax^2 + bx + c = 0$, the sum of the roots equals minus the coefficient of x divided by the coefficient of x^2, and the product of the roots equals the independent term divided by the coefficient of x^2.**

217. Formation of Quadratic Equations.

By aid of the principles of § 216, a quadratic equation may be formed which shall have any required roots.

For, let r_1 and r_2 denote the roots of the equation

$$ax^2 + bx + c = 0, \text{ or } x^2 + \frac{bx}{a} + \frac{c}{a} = 0. \quad (1)$$

Then by § 216, $$\frac{b}{a} = -r_1 - r_2, \text{ and } \frac{c}{a} = r_1 r_2.$$

Substituting these values in (1), we have

$$x^2 - r_1 x - r_2 x + r_1 r_2 = 0.$$

Or, $$(x - r_1)(x - r_2) = 0.$$

Therefore, to form a quadratic equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

Ex. Form the quadratic whose roots shall be 4 and $-\frac{7}{4}$.

By the rule, $$(x - 4)(x + \tfrac{7}{4}) = 0.$$

Multiplying by 4, $$(x - 4)(4x + 7) = 0; \text{ or, } 4x^2 - 9x - 28 = 0.$$

DISCUSSION OF GENERAL EQUATION

218. The roots of a quadratic equation may take several forms:

1. The roots may be rational, unequal, of the same sign.
2. The roots may be rational, unequal, of opposite sign.
3. The roots may be rational, equal.
4. The roots may be irrational, unequal.
5. The roots may be irrational, equal.
6. The roots may be irrational and the number under the radical sign *negative.*

These forms and the causes for their existence are at once seen when one considers the formula in § 213.

By § 213, the roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We will now discuss these results for all possible real values of a, b, and c.

I. $b^2 - 4ac$ *positive.*

In this case, r_1 and r_2 are *real* and *unequal.*

II. $b^2 - 4ac = 0$.

In this case, r_1 and r_2 are *real* and *equal.*

III. $c = 0$.

In this case, the equation takes the form

$$ax^2 + bx = 0; \text{ whence } x = 0 \text{ or } -\frac{b}{a}.$$

Hence, the roots are both *real,* one being *zero.*

IV. $b = 0$, *and* $c = 0$.

In this case, the equation takes the form $ax^2 = 0$.

Hence, both roots equal zero.

V. $b^2 - 4ac$ *negative.*

In this case, r_1 and r_2 are *imaginary* (§ 196).

VI. $b = 0$.

In this case, the equation takes the form

$$ax^2 + c = 0; \text{ whence, } x = \pm\sqrt{-\frac{c}{a}}.$$

If a and c are of unlike sign, the roots are *real, equal in absolute value,* and *unlike in sign.*

If a and c are of like sign, both roots are *imaginary.*

The roots are both *rational,* or both *irrational,* according as $b^2 - 4ac$ is, or is not, a perfect square.

219. It is evident that irrational roots, whether real or imaginary, must occur in *conjugate* pairs.

That is, in an equation of the form of $ax^2 + bx + c = 0$, where a, b, c are real, if one root is of the form $k + \sqrt{h}$ the other must be $k - \sqrt{h}$ where k and h are real.

EXERCISE 46

Find by inspection the sum and product of the roots of the following:

1. $x^2 - 2x - 35 = 0$.
2. $x^2 + 15x + 36 = 0$.
3. $2x^2 + 7x - 4 = 0$.
4. $5x^2 - 13x = -6$.
5. $x^2 + ax - bx = ab$.
6. $cdx^2 + d^2x = c^2x + cd$.
7. $x^2 - 2\sqrt{2}x - 2 = 0$.
8. One root of $8x^2 - 2x - 15 = 0$ is $-1\frac{1}{4}$; find the other.
9. One root of $6x^2 + 11x - 2 = 0$ is $\frac{1}{6}$; find the other.
10. One root of $2x^3 - 8x^2 + 2x + 12 = 0$ is 2; find the others.
11. One root of $m^3 - 7m + 6 = 0$ is -3; find the others.
12. If r_1 and r_2 are the roots of $x^2 + x + 1 = 0$, what does $r_1^2 + r_2^2$ equal? $r_1^3 + r_2^3$?

Form the equations whose roots are:

13. 2, 3.
14. -1, 4.
15. $\frac{1}{2}$, -3.
16. $-\frac{3}{5}$, $-\frac{2}{7}$.
17. 0, $-\frac{5}{6}$.
18. a, $6a$.
19. $a + \sqrt{b}$, $a - \sqrt{b}$.
20. $2 + \sqrt{-3}$, $2 - \sqrt{-3}$.
21. $3c - d$, $-2c + 5d$.

22. $\dfrac{\sqrt{2k}-5\sqrt{g}}{2}, \quad \dfrac{\sqrt{2k}+5\sqrt{g}}{2}.$

23. $6, -\frac{1}{4}, 0.$

Determine by inspection the nature of the roots of the following:

24. $x^2+7x+12=0.$

25. $x^2+8x=-16.$

26. $x^2+2x-1=0.$

27. $x^2+2x+3=0.$

28. $2x^2+7x=3.$

29. $4x^2-16=0.$

30. $2x^2=15x+18.$

31. $x^2-x=12.$

32. $10x^2-x=2.$

33. $23x-6=7x^2.$

34. $16x^2+24x+9=0.$

35. $5x^2+3x=-2.$

GRAPHS

220. The nature of the roots discussed in § 218 is illustrated by the use of graphs:

FIG. 1

x	y
0	-8
1	-9
2	-8
3	-5
4	0
5	7
-1	-5
-2	0
-3	7

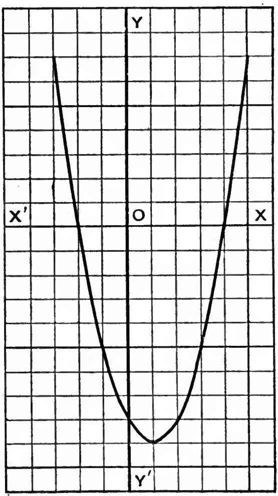


FIG. 1. $x^2-2x-8=0$
$b^2-4ac>0$

FIG. 2

x	y
0	1
1	0
2	1
3	4
4	9
5	16
-2	9
-3	16

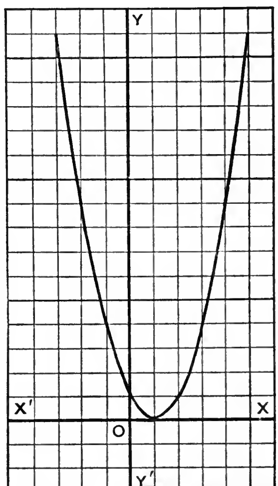
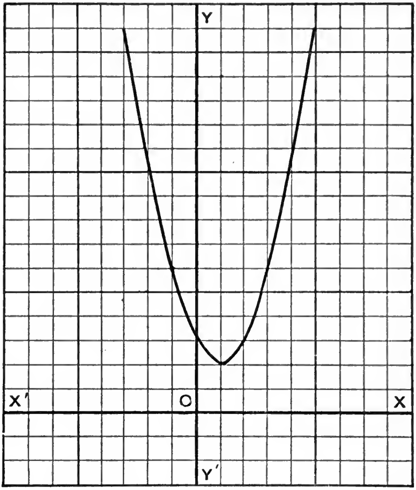


FIG. 2. $x^2-2x+1=0$
$b^2-4ac=0$

FIG. 3

x	y
0	3
1	2
2	3
3	6
4	11
-1	6
-2	11
-3	18



3. $x^2 - 2x + 3 = 0$

$$b^2 - 4ac < 0$$

In Fig. 1, the curve crosses the x-axis at points whose abscissas are 4, -2; the abscissas of these points being the values of x found in solving the equation. In Fig. 2, the intersection points coincide and we have two values of x each equal to 1. In Fig. 3, the curve and the x-axis do not coincide.

EXERCISE 47

Plot the curves:

1. $f(x) = x^2 + 6x + 8$. (§§ 47, 220.)
2. $f(x) = x^2 - 6x + 8$.
3. $f(x) = x^2 - 9$.
4. $f(x) = x^2 - 6x + 9$.
5. $f(x) = x^2 + 2x + 4$.

221. Many problems in Physics are dependent on the laws of proportion and variation. The solution of such problems is often obtained more readily by graphical means than by algebraic solution.

Ex. 1. *Graphical representation of a direct proportion.*

When a man is running at a constant speed, the distance which he travels in a given time is directly proportional to his speed. The algebraic expression of this relation is $\frac{d_1}{d_2} = \frac{s_1}{s_2}$, or $d = ms$. (See § 161.)

Now, if we plot successive values of the distance, d, which correspond to various speeds, s, in precisely the same manner in which we plotted successive values of x and y in §§ 44–48, we obtain as the graphical picture of the relation between s and d a straight line passing through the origin. (See Fig. 1.)

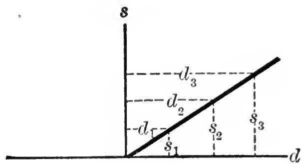


FIG. 1.

This is the graph of any direct proportion.

Ex. 2. *Graphical representation of an inverse proportion.*

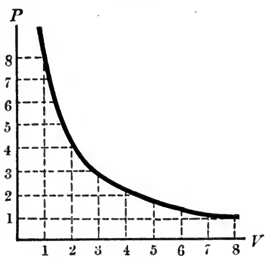


FIG. 2.

The volume which a given body of gas occupies when the pressure to which it is subjected varies has been found to be inversely proportional to the pressure under which the gas stands; we have seen that the algebraic statement of this relation is $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}, \text{ or } V = \frac{m}{P}.$$

$V = 1$, $P = m$. $V = 5$, $P = \frac{m}{5}$.

$V = 2$, $P = \frac{m}{2}$. $V = 6$, $P = \frac{m}{6}$.

$V = 3$, $P = \frac{m}{3}$. $V = 7$, $P = \frac{m}{7}$.

$V = 4$, $P = \frac{m}{4}$. $V = 8$, $P = \frac{m}{8}$.

If we plot successive values of V and P in the manner indicated in §§ 44–48, we obtain a graph of the form shown in Fig. 2.

This is the *graphical representation of any inverse proportion*; the curve is called an **equilateral hyperbola.**

Ex. 3. *The path traversed by a falling body projected horizontally.*

When a body is thrown horizontally from the top of a tower, if it were not for gravity, it would move on in a horizontal direction indefinitely, traversing exactly the same distance in each succeeding second.

Hence, if V represents the velocity of projection, the horizontal distance, H, which it would traverse in any number of seconds, t, would be given by the equation $H = Vt$.

On account of gravity, however, the body is pulled downward, and traverses in this direction in any number of seconds a distance which is given by the equation $S = \frac{1}{2} gt^2$.

To find the actual path taken by the body, we have only to plot successive values of H and S, in the manner in which we plotted the successive values of x and y, in §§ 44–48.

Thus, at the end of 1 second the vertical distance S_1 is given by $S_1 = \frac{1}{2} g \times 1^2 = \frac{1}{2} g$; at the end of 2 seconds we have, $S_2 = \frac{1}{2} g \times 2^2 = \frac{4}{2} g$; at the end of 3 seconds, $S_3 = \frac{1}{2} g \times 3^2 = \frac{9}{2} g$; at the end of 4 seconds, $S_4 = \frac{1}{2} g \times 4^2 = \frac{16}{2} g$; etc.

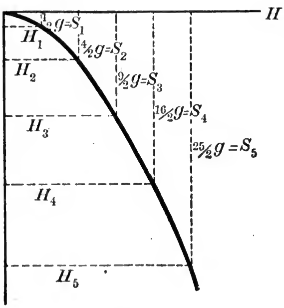


Fig. 3.

On the other hand, at the end of 1 second we have $H_1 = V$; at the end of 2 seconds, $H_2 = 2V$; at the end of 3 seconds, $H_3 = 3V$; at the end of 4 seconds, $H_4 = 4V$.

If, now, we plot these successive values of H and S, we obtain the graph shown in Fig. 3.

This is the path of the body; it is a parabola. (§ 226, Ex. 2.)

Ex. 4. Graph of relation between the temperature and pressure existing within an air-tight boiler containing only water and water vapor.

One use of graphs in physics is to express a relation which is found by experiment to exist between two quantities, which cannot be represented by any simple algebraic equation.

For example, when the temperature of an air-tight boiler which contains only water and water vapor is raised, the pressure within the boiler increases also; thus we find by direct experiment that when the temperature of the boiler is 0° centigrade, the pressure which the vapor exerts will support a column of mercury 4.6 millimeters high.

When the temperature is raised to 10°, the mercury column rises to 9.1 millimeters; at 30° the column is 31.5 millimeters long, etc.

To obtain a simple and compact picture of the relation between temperature and pressure, we plot a succession of temperatures, *e.g.* 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°, 100°, in the manner in which we plotted successive values of x in §§ 44–48, and then plot the corresponding values of pressure obtained by experiment in the manner in which we plotted the y's in §§ 44–48; we obtain the graph shown in Fig. 4.

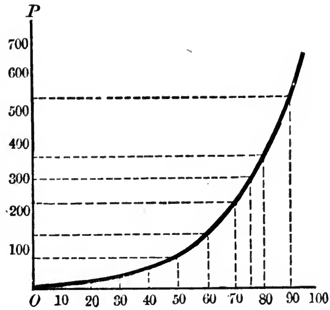


FIG. 4.

From this graph we can find at once the pressure which will exist within the boiler at any temperature.

For example, if we wish to know the pressure at 75° centigrade, we observe where the vertical line which passes through 75° cuts the curve and then run a horizontal line from this point to the point of intersection with the line OP.

This point is found to be at 288; hence the pressure within the boiler at 75° centigrade is 288 millimeters.

EXERCISE 48

PROBLEMS IN PHYSICS

1. When the force which stretches a spring, a straight wire, or any elastic body is varied, it is found that the displacement produced in the body is always directly proportional to the force which acts upon it; *i.e.* if d_1 and d_2 represent any two displacements, and f_1 and f_2 respectively the forces which produce them, then the algebraic statement of the above law is

$$\frac{d_1}{d_2}=\frac{f_1}{f_2}. \tag{1}$$

If a force of 2 pounds stretches a given wire .01 inch, how much will a force of 20 pounds stretch the same wire?

2. If the same force is applied to two wires of the same length and material, but of different diameters, D_1 and D_2, then

the displacements d_1 and d_2 are found to be inversely proportional to the squares of the diameters, *i.e.*

$$\frac{d_1}{d_2}=\frac{D_2^2}{D_1^2}. \qquad (2)$$

If a weight of 100 kilograms stretches a wire .5 millimeter in diameter through 1 millimeter, how much elongation will the same weight produce in a wire 1.5 millimeters in diameter?

3. If the same force is applied to two wires of the same diameter and material, but of different lengths, l_1 and l_2, then it is found that

$$\frac{d_1}{d_2}=\frac{l_1}{l_2}. \qquad (3)$$

From (1), (2), and (3) and § 164, it follows that when lengths, diameters, and forces are all different,

$$\frac{d_1}{d_2}=\frac{f_1}{f_2}\times\frac{l_1}{l_2}\times\frac{D_2^2}{D_1^2}. \qquad (4)$$

If a force of 1 pound will stretch an iron wire which is 200 centimeters long and .5 millimeter in diameter through 1 millimeter, what force is required to stretch an iron wire 150 centimeters long and 1.25 millimeters in diameter through .5 millimeter?

4. When the temperature of a gas is constant, its volume is found to be inversely proportional to the pressure to which the gas is subjected, *i.e.*, algebraically stated,

$$\frac{V_1}{V_2}=\frac{P_2}{P_1}. \qquad (5)$$

At the bottom of a lake 30 meters deep, where the pressure is 4000 grams per square centimeter, a bubble of air has a volume of 1 cubic centimeter as it escapes from a diver's suit. To what volume will it have expanded when it reaches the surface where the atmospheric pressure is about 1000 grams per square centimeter?

5. The electrical resistance of a wire varies directly as its length and inversely as its area. If a copper wire 1 centimeter

in diameter has a resistance of 1 unit per mile, how many units of resistance will a copper wire have which is 500 feet long and 3 millimeters in diameter?

6. The illumination from a source of light varies inversely as the square of the distance from the source. A book which is now 10 inches from the source is moved 15 inches farther away. How much will the light received be reduced?

7. The period of vibration of a pendulum is found to vary directly as the square root of its length. If a pendulum 1 meter long ticks seconds, what will be the period of vibration of a pendulum 30 centimeters long?

8. The force with which the earth pulls on any body outside of its surface is found to vary inversely as the square of the distance from its center. If the surface of the earth is 4000 miles from the center, what would a pound weight weigh 15000 miles from the earth?

9. The number of vibrations made per second by a guitar string of given diameter and material is inversely proportional to its length and directly proportional to the square root of the force with which it is stretched. If a string 3 feet long, stretched with a force of 20 pounds, vibrates 400 times per second, find the number of vibrations made by a string 1 foot long, stretched by a force of 40 pounds.

FACTORING

In Type V, § 103, we learned to transform certain trinomials into Type I, § 103. By means of the results of § 213, we are now able to extend this method to expressions not readily factored by the simpler processes.

222. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2 + bx + c.$$

We have,

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$$

$$= a\left[x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right),$$

by § 103, I.

But by § 213, the roots of $ax^2 + bx + c = 0$ are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Hence, to factor a quadratic expression place it equal to zero, and solve the equation thus formed.

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second.

Sometimes the expression may be written as the difference of two squares and the method of § 103, V, used.

Ex. Factor $x^4 + 1$.

$$x^4 + 1 = (x^4 + 2x^2 + 1) - 2x^2$$
$$= (x^2 + 1)^2 - (x\sqrt{2})^2$$
$$= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1).$$

EXERCISE 49

Factor the following:

1. $x^2 + 11x + 24$.
2. $m^2 - m - 210$.
3. $3a^2 - 10a - 8$.
4. $2a^2 - 11a + 15$.
5. $28 - 13x - 6x^2$.
6. $8c^2 + 8c - 6$.
7. $9x^2 - 6x - 8$.
8. $16 + 18b - 9b^2$.
9. $2 + 5p - 25p^2$.
10. $49a^2 + 28a - 11$.
11. $a^4 + 4$.
12. $x^4 + y^4$.
13. $9d^4 + 22d^2 + 25$.
14. $16a^4 - 78a^2b^2 + 81b^4$.

15. $2x^2 - 5xy + 3y^2 + 5x - 7y + 2.$

16. $x^2 - xy - 2y^2 - 5x + y + 6.$

17. $a^2 + 2ab - 15b^2 - 3ac + 17bc - 4c^2.$

18. $3a^2 - 23ab + 14b^2 + a + 31b - 10.$

19. $6x^2 + 7xy + 2y^2 - 26x - 16y + 24.$

20. $10x^2 + xy - 24y^2 + 26x + 54y - 12.$

Solve the following equations:

21. $x^3 + 27 = 0.$

22. $x^4 - 20x^2 + 64 = 0.$

23. $x^4 + 2x^2 + 9 = 0.$

24. $x^4 + 4x^2 + 9 = 0.$

25. $(x+2)(3x^2+4x+5)=0.$

26. $x^6 - 64 = 0.$

27. $2x^3 - 3x^2 + 4x - 6 = 0.$

28. $x^4 - 2x^3 + 5x^2 = 0.$

29. $(x^2-4)(3x^2+x-10)=0.$

30. $x^7 - 729x = 0.$

31. $x^4 - 9x^2 + 14 = 0.$

32. $9x^4 - 2x^2 + 4 = 0.$

33. $x^4 - 16 = 0.$

34. $2x^3 + 6x^2 - 18x - 54 = 0.$

35. $(4x^2 - 1)(x^2 + x + 1) = 0.$

SIMULTANEOUS QUADRATIC EQUATIONS

223. In solving simultaneous quadratic equations involving two unknown numbers it is necessary to eliminate one of the unknowns as was done in simultaneous linear equations.

The elimination of an unknown number from two equations of the second degree will often produce an equation of the *fourth* degree with one unknown number which cannot be solved by the ordinary methods. The following general directions will lead to the solution of many types.

224. CASE I. *When each equation is in the form*

$$ax^2 + by^2 = c.$$

In this case, either x^2 or y^2 can be eliminated by addition or subtraction (§ 42, II, III).

CASE II. *When each equation is of the second degree, and homogeneous; that is, when each term involving the unknown numbers is of the second degree with respect to them* (§ 23).

The equations may then be solved as follows:

Ex. Solve the equations $\begin{cases} x^2 - 2xy = 5, & (1) \\ x^2 + y^2 = 29. & (2) \end{cases}$

Dividing (1) by (2), $\frac{x^2 - 2xy}{x^2 + y^2} = \frac{5}{29}$, or $29x^2 - 58xy = 5x^2 + 5y^2$.

Then, $5y^2 + 58xy - 24x^2 = 0$, or $(5y - 2x)(y + 12x) = 0$.

Placing $5y - 2x = 0$, $y = \frac{2x}{5}$; substituting in (1),

$$x^2 - \frac{4x^2}{5} = 5, \text{ or } x^2 = 25.$$

Then, $x = \pm 5$, and $y = \frac{2x}{5} = \pm 2$.

Case III. *When the given equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equation.*

Equations of this kind may be solved by combining them in such a way as to obtain the values of $x + y$ and $x - y$.

1. Solve the equations $\begin{cases} x + y = 2. & (1) \\ xy = -15. & (2) \end{cases}$

Squaring (1),	$x^2 + 2xy + y^2 = 4.$	
Multiplying (2) by 4,	$4xy = -60.$	
Subtracting,	$x^2 - 2xy + y^2 = 64.$	
Extracting square roots,	$x - y = \pm 8.$	(3)
Adding (1) and (3),	$2x = 2 \pm 8 = 10$ or $-6.$	
Whence,	$x = 5$ or $-3.$	
Subtracting (3) from (1),	$2y = 2 \mp 8 = -6$ or $10.$	
Whence,	$y = -3$ or $5.$	

The solution is $x = 5,\ y = -3$; or, $x = -3,\ y = 5$.

The above method offers the most desirable form of solution and should be employed when possible.

If one equation is of the second degree, the other of the first degree, and they are not symmetrical, Case IV should be used.

CASE IV. *When one equation is of the second degree and the other of the first.*

Equations of this kind may be solved by finding one of the unknown numbers in terms of the other from the first degree equation, and substituting this value in the other equation.

Ex. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $2y = 7 - x$, or $y = \frac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\frac{7-x}{2}\right) = 6\left(\frac{7-x}{2}\right)$.

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$, or $5x^2 - x = 42$.

Solving, $x = 3$ or $-\frac{14}{5}$.

Substituting in (3), $y = \frac{7-3}{2}$ or $\frac{7+\frac{14}{5}}{2} = 2$ or $\frac{49}{10}$.

The solution is $x = 3$, $y = 2$; or $x = -\frac{14}{5}$, $y = \frac{49}{10}$.

Certain examples where one equation is of the *third* degree and the other of the first may be solved by the method of Case IV.

225. Special Methods for the Solution of Simultaneous Equations of Higher Degree.

No general rules can be given for examples which do not come under the cases just considered; various artifices are employed, familiarity with which can only be gained by experience.

1. Solve the equations $\begin{cases} x^3 - y^3 = 19. & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$

Multiply (2) by 3, $3x^2y - 3xy^2 = 18$. (3)

Subtract (3) from (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 1$.

Extracting cube roots, $x - y = 1$. (4)

Dividing (2) by (4), $xy = 6$. (5)

Solving equations (4) and (5) by the method of § 224, Case III, we find $x = 3$, $y = 2$; or, $x = -2$, $y = -3$.

2. Solve the equations $\begin{cases} x^3 + y^3 = 9\,xy. \\ x + y = 6. \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$(u+v)^3 + (u-v)^3 = 9\,(u+v)(u-v)$, or, $2\,u^3 + 6\,uv^2 = 9\,(u^2 - v^2)$; (1)

and $(u + v) + (u - v) = 6$, $2\,u = 6$, or $u = 3$.

Putting $u = 3$ in (1), $54 + 18\,v^2 = 9\,(9 - v^2)$.

Whence, $v^2 = 1$, or $v = \pm 1$.

Therefore, $x = u + v = 3 \pm 1 = 4$ or 2;

and $y = u - v = 3 \mp 1 = 2$ or 4.

The solution is $x = 4$, $y = 2$; or, $x = 2$, $y = 4$.

The artifice of substituting $u + v$ and $u - v$ for x and y is advantageous in any case where the given equations are *symmetrical* (§ 224, Case III) with respect to x and y. See also Ex. 4.

3. Solve the equations $\begin{cases} x^2 + y^2 + 2\,x + 2\,y = 23. & (1) \\ xy = 6. & (2) \end{cases}$

Multiplying (2) by 2, $2\,xy = 12$. (3)

Add (1) and (3), $x^2 + 2\,xy + y^2 + 2\,x + 2\,y = 35$.

Or, $(x + y)^2 + 2\,(x + y) = 35$.

Completing the square, $(x + y)^2 + 2\,(x + y) + 1 = 36$.

Then, $(x + y) + 1 = \pm 6$; and $x + y = 5$ or -7. (4)

Squaring (4), $x^2 + 2\,xy + y^2 = 25$ or 49.

Multiplying (2) by 4, $4\,xy = 24$.

Subtracting, $x^2 - 2\,xy + y^2 = 1$ or 25.

Whence, $x - y = \pm 1$ or ± 5. (5)

Adding (4) and (5), $2\,x = 5 \pm 1$, or -7 ± 5.

Whence, $x = 3, 2, -1$, or -6.

Subtracting (5) from (4), $2\,y = 5 \mp 1$, or -7 ∓ 5.

Whence, $y = 2, 3, -6$, or -1.

The solution is $x = 3$, $y = 2$; $x = 2$, $y = 3$; $x = -1$, $y = -6$; or $x = -6$, $y = -1$.

4. Solve the equations $\begin{cases} x^4 + y^4 = 97. \\ x + y = -1. \end{cases}$

Putting $x = u + v$ and $y = u - v$,

$$(u+v)^4 + (u-v)^4 = 97, \text{ or } 2\,u^4 + 12\,u^2v^2 + 2\,v^4 = 97, \qquad (1)$$

and $\quad (u+v) + (u-v) = -1,\ 2\,u = -1, \text{ or } u = -\tfrac{1}{2}.$

Substituting value of u in (1), $\tfrac{1}{8} + 3\,v^2 + 2\,v^4 = 97.$

Solving this, $\quad v^2 = \dfrac{25}{4} \text{ or } -\dfrac{31}{4}$; and $v = \pm\dfrac{5}{2} \text{ or } \pm\dfrac{\sqrt{-31}}{2}.$

Then, $x = u + v = -\dfrac{1}{2} \pm \dfrac{5}{2}, \text{ or } -\dfrac{1}{2} \pm \dfrac{\sqrt{-31}}{2} = 2,\ -3, \text{ or } \dfrac{-1 \pm \sqrt{-31}}{2}$;

and $\quad y = u - v = -\dfrac{1}{2} \mp \dfrac{5}{2}, \text{ or } -\dfrac{1}{2} \mp \dfrac{\sqrt{-31}}{2} = -3,\ 2, \text{ or } \dfrac{-1 \mp \sqrt{-31}}{2}.$

The solution is $x = 2,\ y = -3$; $x = -3,\ y = 2$; $x = \dfrac{-1 + \sqrt{-31}}{2}$, $y = \dfrac{-1 - \sqrt{-31}}{2}$; or $x = \dfrac{-1 - \sqrt{-31}}{2}$, $y = \dfrac{-1 + \sqrt{-31}}{2}$.

MISCELLANEOUS EXAMPLES

EXERCISE 50

Solve the following equations and verify each result:

1. $\begin{cases} 2\,xy + x = -36. \\ xy - 3\,y = -5. \end{cases}$

2. $\begin{cases} x^2 + y^2 + x - y = 32. \\ xy = 6. \end{cases}$

3. $\begin{cases} g^2 + h^2 = \frac{289}{36}. \\ gh = \frac{10}{3}. \end{cases}$

4. $\begin{cases} x^2 - 3\,xy - 4\,y^2 = 0. \\ 3\,x - 5\,y = 46. \end{cases}$

5. $\begin{cases} x^2 - 2\,y^2 + 3\,x = -8. \\ x^2 - 2\,y^2 - 4\,y = -2. \end{cases}$

6. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 74. \\ \dfrac{1}{x} - \dfrac{1}{y} = 12. \end{cases}$

7. $\begin{cases} \dfrac{2\,x}{y} - 5\,x = \dfrac{11}{2}. \\ \dfrac{3\,y}{x} + 4\,y = \dfrac{2}{3}. \end{cases}$

8. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = -\dfrac{10}{3}. \\ x - y = 1. \end{cases}$

9. $\begin{cases} x - \dfrac{2}{y} = -\dfrac{a}{b}. \\ y + \dfrac{2}{x} = \dfrac{3b}{a}. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{1}{y^2} = 49. \\ \dfrac{1}{x} + \dfrac{1}{y} = 8. \end{cases}$

10. $\begin{cases} x^4 + y^4 = 17. \\ x - y = 3. \end{cases}$

13. $\begin{cases} x + y = 35. \\ \sqrt[3]{x} + \sqrt[3]{y} = 5. \end{cases}$

11. $\begin{cases} 4d + k - 3dk = -6. \\ d - 5k + 2dk = 10. \end{cases}$

14. $\begin{cases} 11x^2 - xy - y^2 = 45. \\ 7x^2 + 3xy - 2y^2 = 20. \end{cases}$

15. $\begin{cases} x^2 + 4xy = 13. \\ 2xy + 9y^2 = 87. \end{cases}$

16. $\begin{cases} x^2y^2 - 24xy + 95 = 0. \\ 3x - 2y = -13. \end{cases}$

17. $\begin{cases} 3x^2 - 5xy = 2a^2 + 13ab - 7b^2. \\ x + y = 3(a - b). \end{cases}$

18. $\begin{cases} \dfrac{3x+2y}{3x-2y} + \dfrac{3x-2y}{3x+2y} = \dfrac{41}{20}. \\ 8y^2 + 3x^2 = 29. \end{cases}$

24. $\begin{cases} 3x^2 - xy - xz = 4. \\ 5x - 2y = 1. \\ 4x + 3z = -5. \end{cases}$

19. $\begin{cases} \dfrac{1}{xy} = 6a^2. \\ x + y = 5axy. \end{cases}$

25. $\begin{cases} x^2y + xy^2 = 56. \\ x + y = -1. \end{cases}$

20. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = -19a^3. \\ \dfrac{1}{x} + \dfrac{1}{y} = -a. \end{cases}$

26. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{19}{6}. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}. \end{cases}$

21. $\begin{cases} e^2 + 9t^2 + 4e = 9. \\ et + 2t = -2. \end{cases}$

27. $\begin{cases} 3x^2 + 3y^2 = 10xy. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{3}. \end{cases}$

22. $\begin{cases} x^3 + y^3 = 2a^3 + 24a. \\ x^2y + xy^2 = 2a^3 - 8a. \end{cases}$

28. $\begin{cases} x^2y + y^2x = 42. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{7}{6}. \end{cases}$

23. $\begin{cases} \sqrt{2x^2 - 9} = 3y + 6. \\ \sqrt{x^4 - 17y^2} = x^2 - 5. \end{cases}$

29. $\begin{cases} 5q^2 + qs - 3s^2 = 27. \\ 4q^2 - 4qs + 3s^2 = 72. \end{cases}$

30. $\begin{cases} y^2 + 4xy - 3y = 42. \\ 2y^2 - xy + 5y = -10. \end{cases}$

31. $\begin{cases} 16x^2y^2 - 104xy = -105. \\ x - y = -2. \end{cases}$

32.* $\begin{cases} x^4 + x^2y^2 + y^4 = 481. \\ x^2 - xy + y^2 = 37. \end{cases}$

33. $\begin{cases} 9x^2 - 13xy - 3x = -123. \\ xy + 4y^2 + 2y = 125. \end{cases}$

* Divide the first equation by the second.

EXERCISE 51

1. Find two numbers whose product is 112 and whose difference is 6.

2. A rectangular field has a perimeter of 104 rods and an area of 4 acres. Find its dimensions.

3. The square of the sum of two numbers minus four times their product equals 49, and the difference of their squares equals 175. What are the numbers?

4. The sum of the cubes of two numbers is 855; and if the sum of the numbers be multiplied by their product, the result will be 840. What are the numbers?

5. There is a number consisting of two digits, the sum of whose squares is 80; and if the sum of the digits be multiplied by 4, the number will be expressed with its digits reversed. What is the number?

6. A man loaned a sum of money at 6 % for a given time and received $240 interest; if he had loaned the same sum for two years longer at the rate represented by the first number of years, he would have received $40 more than at first. Find the time and the amount loaned.

7. If 5 be added to the denominator and subtracted from the numerator of a certain fraction, it will be expressed by its reciprocal; and the difference of the squares of numerator and denominator equals 65. What is the fraction?

8. A number consists of three digits, the second of which is twice the first. The sum of the squares of the digits equals

89, and if 99 be subtracted from the number, the digits will be reversed. What is the number?

9. A man buys two pieces of cloth, each containing as many yards as its price per yard in cents, and he pays $41 for the whole amount. If the prices for the two pieces of cloth had been interchanged, his bill would have been $1 less. How many yards of each did he buy and what was the price per yard?

10. Two squares have together an area of 613 square rods. If the side of the first square were decreased by 6, and that of the second increased by 1, their perimeters would be in the ratio of 2 to 3. Find the side of each square.

11. There are two numbers whose sum decreased by the square root of their product is 13; and the sum of their squares increased by their product is 481. Find the numbers.

12. Two boys count their pennies. They find that the product of the numbers representing them is 84, and that the square of their sum decreased by twice their difference is 351. How many did each have?

13. There are two numbers whose difference is 819, and the difference of their cube roots is 3. What are the numbers?

14. There is a difference of one hour's time in two trains which go from A to B, the rate of the first train being 5 miles an hour more than that of the second train. If the speed of each train were increased 2 miles per hour, the difference in time from A to B would be decreased 7 minutes 55 seconds. Find the distance from A to B and the rate of each train.

15. The difference of the perimeters of a square and a circle is 5.752 feet and the circle contains 81.86 square feet more than the square. Find the radius of the circle and the side of the square.

16. In an isosceles triangle the product of the base and one leg is 168, and the difference between the squares of the base and leg is 52. Find the altitude of the triangle.

17. The perimeter of a rectangle is 46 inches. If its length be increased 3 inches, its area will be 153 square inches. Find its dimensions. Is there more than one such rectangle? Explain.

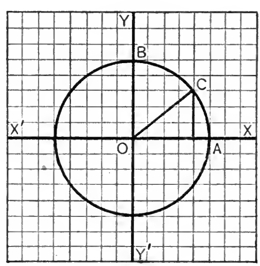
18. If the sum of the denominator and numerator of a certain fraction be divided by their difference, the quotient is 9. But if the product of the numerator and denominator be divided by their sum, the quotient is 2 with a remainder of 2. Find the fraction. What principle of proportion is illustrated in this problem? If this principle is applied, are simultaneous equations necessary?

226. It was noted in §§ 224, 225, that two second degree equations had four solutions, or pairs of values for x and y, that a second degree and a first degree equation had two solutions, that if imaginary roots entered they were always in pairs. The geometric explanation for this is readily seen if the equations are plotted.

Ex. 1. Consider the equation $x^2 + y^2 = 25$.

This means that, for any point on the graph, the square of the abscissa, plus the square of the ordinate, equals 25.

But the square of the abscissa of any point, plus the square of the ordinate, equals the square of the distance of the point from the origin; for the distance is the hypotenuse of a right triangle, whose other two sides are the abscissa and ordinate.

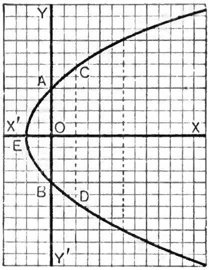


Then the square of the distance from O of any point on the graph is 25; or, the distance from O of any point on the graph is 5.

Thus, the graph is a circle of radius 5, having its center at O.

(The graph of any equation of the form $x^2 + y^2 = a$ is a circle.)

Ex. 2. Consider the equation $y^2 = 4x + 4$.



If $x = 0$, $y^2 = 4$, or $y = \pm 2$. (A, B)

If $x = 1$, $y^2 = 8$, or $y = \pm 2\sqrt{2}$. (C, D)

If $x = -1$, $y = 0$, Etc. (E)

The graph extends indefinitely to the right of YY'.

If x is negative and < -1, y^2 is negative, and therefore y *imaginary;* then, no part of the graph lies to the left of E.

(The graph of Ex. 2 is a *parabola;* as also is the graph of any equation of the form $y^2 = ax$ or $y^2 = ax + b$.)

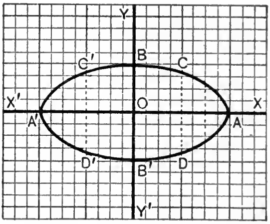
Ex. 3. Consider the equation $x^2 + 4y^2 = 4$.

In this case it is convenient to first locate the points where the graph intersects the axes.

If $y = 0$, $x^2 = 4$, or $x = \pm 2$. (A, A')

If $x = 0$, $4y^2 = 4$, or $y = \pm 1$. (B, B')

Putting $x = \pm 1$, $4y^2 = 3$, $y^2 = \frac{3}{4}$, or $y = \pm \frac{\sqrt{3}}{2}$. ($C$, D, C', D')

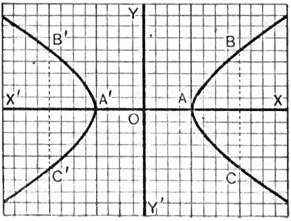


If x has any value > 2, or < -2, y^2 is negative, and y imaginary; then, no part of the graph lies to the right of A, or left of A'.

If y has any value > 1, or < -1, x^2 is negative, and x imaginary; then, no part of the graph lies above B, or below B'.

(The graph of Ex. 3 is an *ellipse;* as also is the graph of any equation of the form $ax^2 + by^2 = c$.)

Ex. 4. Consider the equation $x^2 - 2y^2 = 1$.



Here $x^2 - 1 = 2y^2$, or $y^2 = \frac{x^2 - 1}{2}$.

If $x = \pm 1$, $y^2 = 0$, or $y = 0$. (A', A)

If x has any value between 1 and -1, y^2 is negative, and y imaginary. Then, no part of the graph lies between A and A'.

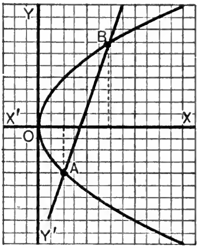
If $x = \pm 2$, $y^2 = \frac{3}{2}$, or $y = \pm\sqrt{\frac{3}{2}}$. ($B$, C, B', C')

The graph has two branches BAC and $B'A'C'$, each of which extends to an indefinitely great distance from O.

(The graph of Ex. 4 is a *hyperbola;* as also is the graph of any equation of the form $ax^2 - by^2 = c$, or $xy = a$.)

227. Graphical Representation of Solutions of Simultaneous Quadratic Equations.

Ex. 1. Consider the equations $\begin{cases} y^2 = 4x, \\ 3x - y = 5. \end{cases}$



The graph of $y^2 = 4x$ is the parabola AOB.

The graph of $3x - y = 5$ is the straight line AB, intersecting the parabola at the points A and B, respectively.

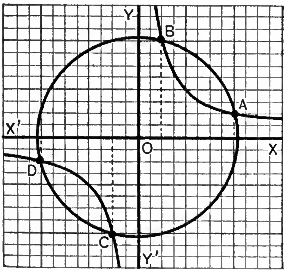
To find the coördinates of A and B, we proceed as in § 48; that is, we *solve the given equations.*

The solution is $x = 1$, $y = -2$; or, $x = \frac{25}{9}$, $y = \frac{10}{3}$ (§ 224, IV).

It may be verified in the figure that these are the coördinates of A and B, respectively.

Hence, *if any two graphs intersect, the coördinates of any point of intersection form a solution of the set of equations represented by the graphs.*

Ex. 2. Consider the equations $\begin{cases} x^2 + y^2 = 17, \\ xy = 4. \end{cases}$



The graph of $x^2 + y^2 = 17$ is the circle AD, whose centre is at O, and radius $\sqrt{17}$.

The graph of $xy = 4$ is a hyperbola, having its branches in the angles XOY and $X'OY'$, respectively, and intersecting the circle at the points A and B in angle XOY, and at the points C and D in angle $X'OY'$.

The solution of the given equation is

$$x = 4,\ y = 1;\ x = 1,\ y = 4;$$
$$x = -1,\ y = -4;\ \text{and}\ x = -4,\ y = -1.$$

It may be verified in the figure that these are the coördinates of A, B, C, and D, respectively.

EXERCISE 52

Find the graphs of the following sets of equations, and in each case verify the principle of § 227:

1. $\begin{cases} x^2 + 4y^2 = 4. \\ x - y = 1. \end{cases}$

2. $\begin{cases} x^2 - 4y = -7. \\ 2x + 3y = 4. \end{cases}$

3. $\begin{cases} 9x^2 + y^2 = 148. \\ xy = -8. \end{cases}$

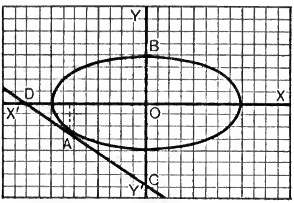
4. $\begin{cases} x^2 + y^2 = 29. \\ xy = 10. \end{cases}$

5. $\begin{cases} 2x^2 + 5y^2 = 53. \\ 3x^2 - 4y^2 = -24. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 13. \\ 4x - 9y = 6. \end{cases}$

228. *Ex.* 1. Consider the equations

$$\begin{cases} x^2 + 4y^2 = 4, & (1) \\ 2x + 3y = -5. & (2) \end{cases}$$



Scale: $\frac{1}{4}$ inch.

The graph of $x^2 + 4y^2 = 4$ is the ellipse AB.

The graph of $2x + 3y = -5$ is the straight line CD.

If y or x is eliminated between these two equations, we find that the resulting equation containing one unknown number is such that if all the terms are transposed to one member, that member is a trinomial perfect square. Hence, the equation has equal roots and the line and curve are tangent at A (218, II).

If in Ex. 1, § 228, the second equation had been $2x + 3y = -10$ (2), the roots would have been imaginary and the line would not have met the ellipse.

REVIEW EXAMPLES

EXERCISE 53

1. Reduce $\dfrac{\sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ to the form $\dfrac{1}{1-x^2}$.

2. Reduce $\dfrac{\dfrac{a}{x^2}}{1 + \dfrac{a^2}{x^2}} + \dfrac{\dfrac{a}{(x-a)^2}}{\dfrac{x+a}{x-a}}$.

3. Reduce $\dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$ to the form $\left(\dfrac{2}{e^x + e^{-x}}\right)^2$.

4. Reduce $\dfrac{1}{\sqrt{2ax - x^2}}$ to the form $\dfrac{1}{a\sqrt{2\left(\dfrac{x}{a}\right) - \left(\dfrac{x}{a}\right)^2}}$.

5. Reduce $\dfrac{(x^{\frac{m}{n}} + x^{-\frac{m}{n}})^2 - 4}{x^{\frac{2m}{n}} - x^{-\frac{2m}{n}}}$ to the form $\dfrac{x^{\frac{m}{n}} - x^{-\frac{m}{n}}}{x^{\frac{m}{n}} + x^{-\frac{m}{n}}}$.

6. Reduce $\dfrac{1 + \dfrac{x}{\sqrt{x^2 - a^2}}}{x + \sqrt{x^2 - a^2}} + \dfrac{\dfrac{1}{a}}{\dfrac{x}{a}\sqrt{\dfrac{x^2 - a^2}{a^2}}}$ to the form $\dfrac{1}{x}\sqrt{\dfrac{x + a}{x - a}}$.

7. Reduce $\dfrac{x[1 + x(x^2 + y^2)^{-\frac{1}{2}}]}{x + \sqrt{x^2 + y^2}} + \dfrac{y^2(x^2 + y^2)^{-\frac{1}{2}}}{x + \sqrt{x^2 + y^2}}$ to unity.

8. Reduce $\dfrac{\dfrac{ax}{(a^2 - x^2)^{\frac{3}{2}}}}{\dfrac{a}{(a^2 - x^2)^{\frac{1}{2}}}\sqrt{\left(\dfrac{a}{\sqrt{a^2 - x^2}}\right)^2 - 1}}$ to the form $\dfrac{1}{\sqrt{a^2 - x^2}}$.

9. $S = \sqrt{1 + K^2}$. If $K = \dfrac{\sqrt{2ay - y^2}}{y}$, find $S = \sqrt{\dfrac{2a}{y}}$.

10. Reduce $\dfrac{1}{\sqrt{1 - 3x - x^2}}$ to the form $\dfrac{\dfrac{2}{\sqrt{13}}}{\sqrt{1 - \left(\dfrac{3 + 2x}{\sqrt{13}}\right)^2}}$.

11. Reduce $\dfrac{x^3}{x^8 - x^4 - 6}$ to the form $\dfrac{1}{25}\left[\dfrac{-4x^3}{1 - \left(\dfrac{1 - 2x^4}{5}\right)^2}\right]$.

12. Reduce $\sqrt{2ax - x^2}$ to the form

$$\frac{a}{\sqrt{1 - \left(\dfrac{x - a}{a}\right)^2}} - \frac{(x - a)^2}{a\sqrt{1 - \left(\dfrac{x - a}{a}\right)^2}}.$$

13. $2\tan x + (\tan x)^2 - 3 = 0$; find $\tan x$.

14. $2\cos x + \dfrac{1}{\cos x} = 3$; find $\cos x$.

15. $\dfrac{1}{\cot x} + 2\cot x = \dfrac{5}{2}\sqrt{1+\cot^2 x}$; find $\cot x$.

16. Reduce $\dfrac{n(1+x)^n \cdot x^{n-1} - n(1+x)^{n-1}\cdot x^n}{(1+x)^{2n}}$ to $\dfrac{nx^{n-1}}{(1+x)^{n+1}}$.

17. $S = \dfrac{1}{3\sqrt{3}}(12+x)^{\frac{3}{2}} - 8$. Evaluate S when $x = 15$.

18. Reduce $\dfrac{-x^2\left(\dfrac{4x^3}{2\sqrt{1-x^4}}\right) - 2x - 2x\sqrt{1-x^4}}{x^4}$ to the form

$$-\frac{2}{x^3}\left(1+\frac{1}{\sqrt{1-x^4}}\right).$$

19. Reduce $\dfrac{2+\dfrac{2x-1}{\sqrt{x^2-x-1}}}{2x-1+2\sqrt{x^2-x-1}}$ to the form $\dfrac{1}{\sqrt{x^2-x-1}}$.

20. Reduce $\dfrac{\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}} + \dfrac{3}{4}\left(\dfrac{x-2}{x+2}\right)^{-\frac{1}{4}}\left(\dfrac{2}{x+2}\right)^2}{\left(\dfrac{x-2}{x+2}\right)^{\frac{3}{4}}}$ to the form $\dfrac{x^2-1}{x^2-4}$.

21. Reduce $\dfrac{\left[1+\left(\dfrac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)^2\right]\left(\dfrac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)}{\dfrac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2a}}$ to the form

$$\frac{a\left(e^{\frac{2x}{a}}-e^{-\frac{2x}{a}}\right)}{4}.$$

22. Reduce $y+\dfrac{1+\left(\dfrac{\sqrt{2ry-y^2}}{y}\right)^2}{\dfrac{-r}{y^2}}$ to the form $-y$.

23. Reduce $x-\dfrac{\left(1+\dfrac{4p^2}{y^2}\right)\dfrac{2p}{y}}{\dfrac{-4p^2}{y^3}}$ to $3x+2p$ when $y^2=4px$.

24. Reduce $-\dfrac{b^2}{a^2}\left[\dfrac{y-x\left(\dfrac{-b^2x}{a^2y}\right)}{y^2}\right]$ to $-\dfrac{b^4}{a^2y^3}$ when $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

IX. SERIES

ARITHMETIC PROGRESSION

229. An **Arithmetic Progression** is a series of terms in which each term, after the first, is obtained by adding to the preceding term a constant number called the *Common Difference.*

Thus, 1, 3, 5, 7, 9, 11, ⋯ is an arithmetic progression in which the common difference is 2.

Again, 12, 9, 6, 3, 0, − 3, ⋯ is an arithmetic progression in which the common difference is − 3.

An Arithmetic Progression is also called an *Arithmetic Series.*

230. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is a, $a+d$, $a+2d$, $a+3d$, ⋯.

We observe that the coefficient of d in any term is less by 1 than the number of the term.

Then, in the nth term the coefficient of d will be $n-1$.

That is, $$l=a+(n-1)d. \qquad \text{(I)}$$

231. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the terms, S.*

$$S = a + (a + d) + (a + 2d) + \cdots + (l - d) + l.$$

Writing the terms in reverse order,

$$S = l + (l - d) + (l - 2d) + \cdots + (a + d) + a.$$

Adding these equations term by term,

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l).$$

Therefore, $2S = n(a + l)$, and $S = \frac{n}{2}(a + l)$. (II)

232. Substituting in (II) the value of l from (I), we have

$$S = \frac{n}{2}[2a + (n - 1)d].$$

Ex. In the progression 8, 5, 2, -1, -4, ..., to 27 terms, find the last term and the sum.

Here, $a = 8$, $d = 5 - 8 = -3$, $n = 27$.

Substitute in (I), $l = 8 + (27 - 1)(-3) = 8 - 78 = -70$.

Substitute in (II), $S = \frac{27}{2}(8 - 70) = 27(-31) = -837$.

The common difference may be found by subtracting the first term from the second, or any term from the next following term.

EXERCISE 54

In each of the following find the last term and then the sum:

1. 2, 5, 8, ..., to 17 terms.
2. 3, 9, 15, ..., to 12 terms.
3. 7, 5, 3, ..., to 24 terms.
4. 1, $\frac{1}{2}$, 0, ..., to 32 terms.
5. $-\frac{1}{3}$, $-\frac{1}{12}$, $\frac{1}{6}$, ..., to 9 terms.
6. a, $a - 3b$, $a - 6b$, ..., to 15 terms.
7. $2x + 5y$, $x + 4y$, $3y$, ..., to 13 terms.

8. $\frac{2c-5d}{3}, \frac{c-4d}{6}, \frac{d-c}{3}, \cdots$, to 20 terms.

9. $\frac{3}{5x}, \frac{1}{10x}, -\frac{2}{5x}, \cdots$, to 19 terms.

10. $\frac{1}{25}, \frac{7}{50}, \frac{6}{25}, \cdots$, to 47 terms.

233. The *first term, common difference, number of terms, last term,* and *sum of the terms* are called the *elements* of the progression.

If any three of the five elements of an arithmetic progression are given, the other two may be found by substituting the known values in the fundamental formulæ (I) and (II), and solving the resulting equations.

1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Substituting the given values in (II),

$$-\tfrac{5}{3} = 10\left(-\tfrac{5}{3} + l\right) \text{ or } -\tfrac{1}{6} = -\tfrac{5}{3} + l;\ \text{then, } l = \tfrac{5}{3} - \tfrac{1}{6} = \tfrac{3}{2}.$$

Substituting the values of a, n, and l in (I), $\frac{3}{2} = -\frac{5}{3} + 19d$.

Whence, $19d = \frac{3}{2} + \frac{5}{3} = \frac{19}{6}$, and $d = \frac{1}{6}$.

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (I), $-39 = a + (n-1)(-3)$, or $a = 3n - 42$. (1)

Substituting the values of l, S, and a in (II),

$$-264 = \frac{n}{2}(3n - 42 - 39), \text{ or } -528 = 3n^2 - 81n, \text{ or } n^2 - 27n + 176 = 0.$$

Whence, $n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16$ or 11.

Substituting in (1), $a = 48 - 42$ or $33 - 42 = 6$ or -9.

The solution is $a = 6$, $n = 16$; or, $a = -9$, $n = 11$.

The significance of the two answers is as follows:

If $a = 6$ and $n = 16$, the progression is $6, 3, 0, -3, -6, -9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39$.

If $a = -9$ and $n = 11$, the progression is

$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39$.

In each of these the sum is -264.

3. Given $a=\frac{1}{3}$, $d=-\frac{1}{12}$, $S=-\frac{3}{2}$; find l and n.

Substituting in (I), $l=\frac{1}{3}+(n-1)\left(-\frac{1}{12}\right)=\frac{5-n}{12}$. (1)

Substituting the values of a, S, and l in (II),

$$-\frac{3}{2}=\frac{n}{2}\left(\frac{1}{3}+\frac{5-n}{12}\right), \text{ or } -3=n\left(\frac{9-n}{12}\right), \text{ or } n^2-9n-36=0.$$

Whence, $$n=\frac{9\pm\sqrt{81+144}}{2}=\frac{9\pm 15}{2}=12 \text{ or } -3.$$

The value $n=-3$ must be rejected, for the number of terms in a progression must be a *positive integer*.

Substituting $n=12$ in (1), $l=\frac{5-12}{12}=-\frac{7}{12}$.

A *negative* or *fractional* value of n must be rejected, together with all other values dependent on it.

EXERCISE 55

1. Given $a=5$, $d=2$, $l=65$; find n and S.
2. Given $d=-3$, $n=42$, $l=-119$; find a and S.
3. Given $d=\frac{5}{6}$, $n=16$, $S=\frac{316}{3}$; find a and l.
4. Given $n=19$, $l=\frac{-59}{7}$, $S=-\frac{1045}{14}$; find a and d.
5. Given $S=-540$, $a=-23$, $n=48$; find l and d.
6. Given $d=-4$, $l=\frac{-3325}{64}$, $S=\frac{-11627}{32}$; find a and n.
7. Given $d=a-1$, $a=2a+5$, $S=44a+12$; find l and n.
8. Given $a=-8a$, $l=8a-16b$, $S=-136b$; find n and d.
9. Given $a=.4$, $l=34.6$, $n=20$; find d and S.
10. Given $S=18.15$, $d=.02$, $a=.23$; find l and n.
11. Given $S=\frac{-209}{3}$, $d=\frac{-3}{5}$, $n=15$; find a and l.
12. Given $n=26$, $d=\frac{7}{8}$, $l=\frac{-325}{8}$; find S and a.

234. From (I) and (II), *general formulæ* for the solution of examples like the above may be readily derived.

Ex. Given a, d, and S; derive the formula for n.

By § 232, $2S = n[2a + (n-1)d]$, or $dn^2 + (2a - d)n = 2S$.

This is a quadratic in n, and may be solved by the method of § 213; multiplying by $4d$, and adding $(2a - d)^2$ to both members,

$$4d^2n^2 + 4d(2a - d)n + (2a - d)^2 = 8dS + (2a - d)^2.$$

Extracting square roots, $2dn + 2a - d = \pm\sqrt{8dS + (2a - d)^2}$.

Whence,
$$n = \frac{d - 2a \pm \sqrt{8dS + (2a - d)^2}}{2d}.$$

EXERCISE 56

1. Given a, l, and n; derive the formula for d.
2. Given a, n, and S; derive the formulæ for d and l.
3. Given d, n, and S; derive the formulæ for a and l.
4. Given a, d, and l; derive the formulæ for n and S.
5. Given d, l, and n; derive the formulæ for a and S.
6. Given l, n, and S; derive the formulæ for a and d.
7. Given a, d, and S; derive the formulæ for l.
8. Given a, l, and S; derive the formulæ for d and n.
9. Given d, l, and S; derive the formulæ for a and n.

235. Arithmetic Means.

We define *inserting m arithmetic means between two given numbers, a and b,* as finding an arithmetic progression of $m + 2$ terms, whose first and last terms are a and b.

Ex. Insert 5 arithmetic means between 3 and -5.

We find an arithmetic progression of 7 terms, in which $a = 3$, and $l = -5$; substituting $n = 7$, $a = 3$, and $l = -5$ in (I),

$$-5 = 3 + 6d, \text{ or } d = -\tfrac{4}{3}.$$

The progression is $3, \frac{5}{3}, \frac{1}{3}, -1, -\frac{7}{3}, -\frac{11}{3}, -5$.

236. Let x denote the arithmetical mean between a and b.

Then, $$x - a = b - x, \text{ or } 2x = a + b.$$

Whence, $$x = \frac{a+b}{2}.$$

That is, *the arithmetic mean between two numbers equals one-half their sum.*

EXERCISE 57

1. Insert 6 arithmetic means between 3 and 24.
2. Insert 12 arithmetic means between -5 and 73.
3. Insert 20 arithmetic means between $\frac{4}{5}$ and $-\frac{58}{15}$.
4. Insert 13 arithmetic means between $-\frac{1}{3}$ and $-\frac{23}{6}$.
5. Find the arithmetic mean between $a^2 - 2a - 9$ and $a^2 - 6a + 1$.
6. If $n - 2$ arithmetic means are inserted between a and l, find the 4th term.

GEOMETRIC PROGRESSION

237. A **Geometric Progression** is a series of terms in which each term, after the first, is obtained by multiplying the preceding term by a constant number called the *Ratio.*

Thus, 2, 6, 18, 54, 162, ⋯ is a geometric progression in which the ratio is 3.

9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ⋯ is a geometric progression in which the ratio is $\frac{1}{3}$.

-3, 6, -12, 24, -48, ⋯ is a geometric progression in which the ratio is -2.

A Geometric Progression is also called a *Geometric Series.*

238. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is $a, ar, ar^2, ar^3, \cdots$.

We observe that the exponent of r in any term is less by 1 than the number of the term.

Then, in the nth term the exponent of r will be $n - 1$.

That is, $$l = ar^{n-1}. \qquad \text{(I)}$$

239. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the terms, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying each term by r,

$$rS = ar = ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), $rS - S = ar^n - a$, or $S = \dfrac{ar^n - a}{r-1}$.

But by (I), § 238, $\qquad rl = ar^n.$

Therefore, $$S = \frac{rl - a}{r-1}. \qquad (II)$$

The *first term*, *ratio*, *number of terms*, *last term*, and *sum of the terms* are called the *elements* of the progression.

240. Examples.

1. In the progression 3, 1, $\frac{1}{3}$, ⋯, to 7 terms, find the last term and the sum.

Here, $\qquad a = 3,\ r = \frac{1}{3},\ n = 7.$

Substituting in (I), $\qquad l = 3\left(\frac{1}{3}\right)^6 = \frac{1}{3^5} = \frac{1}{243}.$

Substituting in (II), $\qquad S = \dfrac{\frac{1}{3} \times \frac{1}{243} - 3}{\frac{1}{3} - 1} = \dfrac{\frac{1}{729} - 3}{-\frac{2}{3}} = \dfrac{-\frac{2186}{729}}{-\frac{2}{3}} = \dfrac{1093}{243}.$

The ratio may be found by dividing the second term by the first, or any term by the next preceding term.

2. In the progression -2, 6, -18, ⋯, to 8 terms, find the last term and the sum.

Here, $\qquad a = -2,\ r = \dfrac{6}{-2} = -3,\ n = 8$, therefore,

$$l = -2(-3)^7 = -2 \times (-2187) = 4374.$$

$$S = \frac{-3 \times 4374 - (-2)}{-3-1} = \frac{-13122 + 2}{-4} = 3280.$$

EXERCISE 58

Find the last term and sum of the following:

1. $1, 3, 9, \cdots$ to 8 terms.
2. $2, 1, \frac{1}{2}, \cdots$ to 11 terms.
3. $5, -10, 20, \cdots$ to 12 terms.
4. $-\frac{1}{5}, \frac{2}{15}, -\frac{4}{45}, \cdots$ to 7 terms.
5. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \cdots$ to 6 terms.
6. $-\frac{7}{8}, -\frac{7}{10}, -\frac{14}{25}, \cdots$ to 8 terms.

241. If any three of the five elements of a geometric progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I) and (II), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown number appears as an exponent, the solution of which form of equation can usually only be effected by the aid of logarithms (§ 110).

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I), we have

$$-32 = -2r^4;\ \text{whence, } r^4 = 16, \text{ or } r = \pm 2.$$

Substituting in (II),

If $\quad r = 2,\quad S = \dfrac{2(-32) - (-2)}{2 - 1} = -64 + 2 = -62.$

If $\quad r = -2,\quad S = \dfrac{(-2)(-32) - (-2)}{-2 - 1} = \dfrac{64 + 2}{-3} = -22.$

The solution is $r = 2$, $S = -62$; or, $r = -2$, $S = -22$.

The interpretation of the two answers is as follows:

If $r = 2$, the progression is $-2, -4, -8, -16, -32$, whose sum is -62.

If $r = -2$, the progression is $-2, 4, -8, 16, -32$, whose sum is -22.

2. Given $a=3$, $r=-\frac{1}{3}$, $S=\frac{1640}{729}$; find n and l.

Substituting in (II), $\frac{1640}{729}=\frac{-\frac{1}{3}l-3}{-\frac{1}{3}-1}=\frac{l+9}{4}$.

Whence, $l+9=\frac{6560}{729}$; or, $l=\frac{6560}{729}-9=-\frac{1}{729}$.

Substituting the values of a, r, and l in (I),

$$-\frac{1}{729}=3(-\tfrac{1}{3})^{n-1}; \text{ or, } (-\tfrac{1}{3})^{n-1}=-\frac{1}{2187}.$$

Whence, by inspection, $n-1=7$, or $n=8$.

From (I) and (II) general formulæ may be derived for the solution of cases like the above.

If the given elements are n, l, and S, *equations* for a and r may be found, but there are no definite *formulæ* for their values.

The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in § 110.

EXERCISE 59

1. Given $r=2$, $n=12$, $S=4095$; find a and l.
2. Given $a=2$, $r=-3$, $l=1458$; find n and S.
3. Given $l=-\frac{1}{800}$, $a=-\frac{16}{25}$, $n=10$; find r and S.
4. Given $a=\frac{7}{8}$, $l=3584$, $S=\frac{38227}{8}$; find r and n.
5. Given $r=\frac{1}{3}$, $n=5$, $l=\frac{1}{135}$; find a and S.
6. Given $S=-\frac{4681}{64}$, $a=-64$, $r=\frac{1}{8}$; find n and l.
7. Given a, l, and S; derive the formula for r.
8. Given r, l, and n; derive the formulæ for a and S.
9. Given a, n, and l; derive the formulæ for r and S.
10. Given S, n, and r; derive the formulæ for a and l.

242. Sum of a Geometric Progression to Infinity.

The limit (§ 125) to which the sum of the terms of a *decreasing* geometric progression approaches, when the number of terms is indefinitely increased, is called the *sum of the series to infinity.*

Formula (II), § 239, may be written

$$S = \frac{a - rl}{1 - r}.$$

It is evident that, by sufficiently continuing a decreasing geometric progression, the absolute value of the last term may be made less than any assigned number, however small.

Hence, when the number of terms is indefinitely increased, l, and therefore rl, approaches the limit 0.

Then, the fraction $\frac{a - rl}{1 - r}$ approaches the limit $\frac{a}{1 - r}$.

Therefore, the sum of a decreasing geometric progression to infinity is given by the formula

$$S = \frac{a}{1 - r}. \qquad \text{(III)}$$

Ex. Find the sum of the series 4, $-\frac{8}{3}$, $\frac{16}{9}$, $\cdots$ to infinity.

Here $a = 4$, $r = -\frac{2}{3}$.

Substituting in (III), $S = \frac{4}{1 + \frac{2}{3}} = \frac{12}{5}$.

To find the value of a repeating decimal.

This is a case of finding the sum of a decreasing geometric series to infinity, and may be solved by formula (III).

Ex. Find the value of .85151⋯.

We have, $.85151 \cdots = .8 + .051 + .00051 + \cdots$.

The terms after the first constitute a decreasing geometric progression in which $a = .051$, and $r = .01$.

Substituting in (III), $S = \frac{.051}{1 - .01} = \frac{.051}{.99} = \frac{51}{990} = \frac{17}{330}$.

Then the value of the given decimal is $\frac{8}{10} + \frac{17}{330}$, or $\frac{281}{330}$.

EXERCISE 60

Find the sum to infinity of the following:

1. $2, \frac{2}{3}, \frac{2}{9}, \cdots$.
2. $1, -\frac{1}{2}, \frac{1}{4}, \cdots$.
3. $\frac{5}{6}, \frac{1}{6}, \frac{1}{30}, \cdots$.
4. $-\frac{8}{9}, -\frac{16}{27}, -\frac{32}{81}, \cdots$.
5. $-.3, .12, -.048, \cdots$.
6. $6, -3, \frac{3}{2}, \cdots$.

Find the values of the following:

7. $.4777 \cdots$. 8. $.8181 \cdots$. 9. $.5243243 \cdots$. 10. $.207575 \cdots$.

243. Geometric Means.

We define *inserting m geometric means between two numbers, a and b,* as finding a geometric progression of $m+2$ terms, whose first and last terms are a and b.

Ex. Insert 5 geometric means between 2 and $\frac{128}{729}$.

We find a geometric progression of 7 terms, in which $a = 2$, and $l = \frac{128}{729}$; substituting $n = 7$, $a = 2$, and $l = \frac{128}{729}$ in (I),

$$\frac{128}{729} = 2r^6; \text{ whence } r^6 = \frac{64}{729}, \text{ and } r = \pm \frac{2}{3}.$$

The result is $2, \pm\frac{4}{3}, \frac{8}{9}, \pm\frac{16}{27}, \frac{32}{81}, \pm\frac{64}{243}, \frac{128}{729}$.

244. Let x denote the geometric between a and b.

Then, $$\frac{x}{a} = \frac{b}{x}, \text{ or } x^2 = ab.$$

Whence, $$x = \sqrt{ab}.$$

That is, *the geometric mean between two numbers is equal to the square root of their product.*

245. Problems.

1. The sixth term of an arithmetic progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By § 230, the sixth term is $a + 5d$, and the fifteenth term $a + 14d$.

Then, by the conditions, $$\begin{cases} a + 5d = \frac{5}{6}. & (1) \\ a + 14d = \frac{16}{3}. & (2) \end{cases}$$

Subtracting (1) from (2), $9d = \frac{9}{2}$; whence, $d = \frac{1}{2}$.

Substituting in (1), $a + \frac{5}{2} = \frac{5}{6}$; whence, $a = -\frac{5}{3}$.

2. Find four numbers in arithmetic progression such that the product of the first and fourth shall be 45, and the product of the second and third 77.

Let the numbers be $x - 3y$, $x - y$, $x + y$, and $x + 3y$.

Then by the conditions, $\begin{cases} x^2 - 9y^2 = 45. \\ x^2 - y^2 = 77. \end{cases}$

Solving these equations, $x = 9$, $y = \pm 2$; or, $x = -9$, $y = \pm 2$ (§ 224)
Then the numbers are 3, 7, 11, 15; or, -3, -7, -11, -15.

In problems like the above, it is convenient to represent the unknown numbers by *symmetrical* expressions.

Thus, if five numbers had been required, we should have represented them by $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

3. Find 3 numbers in geometric progression such that their sum shall be 14, and the sum of their squares 84.

Let the numbers be represented by a, ar, and ar^2.

Then, by the conditions, $\begin{cases} a + ar + ar^2 = 14. & (1) \\ a^2 + a^2r^2 + a^2r^4 = 84. & (2) \end{cases}$

Divide (2) by (1), $a - ar + ar^2 = 6.$ (3)

Subtract (3) from (1), $2ar = 8$, or $r = \frac{4}{a}$. (4)

Substituting in (1), $a + 4 + \frac{16}{a} = 14$, or $a^2 - 10a + 16 = 0$.

Solving this equation, $a = 8$ or 2.

Substituting in (4), $r = \frac{4}{8}$ or $\frac{4}{2} = \frac{1}{2}$ or 2.

Then, the members are 2, 4, and 8.

EXERCISE 61

1. The seventh term of an A. P. is $\frac{64}{15}$, the twenty-first term is $\frac{38}{3}$. Find the fifteenth term.

2. Show that the sum of the odd integers from 1 to 999 is the square of their number.

3. The first term of an A. P. is 1, the sum of the third and ninth terms is 32. Find the sum of the first thirteen terms.

4. The sum of the first ten terms of an A. P. is to the sum of the first seven terms as 29 to 14. Find the ratio of the common difference to the first term.

5. There are four numbers, such that the first three form a G. P., the last three form an A. P. The sum of the first three is 73, of the last three 192. The difference between the second and fourth is 112. Find the numbers.

6. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$, when their sum is $\frac{21}{2}$?

7. Find four numbers in A. P., such that the sum of the first and second shall be -1, and the product of the second and fourth 24.

8. A traveller sets out from a certain place, and goes 7 miles the first hour, $7\frac{1}{2}$ the second hour, 8 the third hour, and so on. After he has been gone 5 hours, another sets out and travels $16\frac{1}{4}$ miles an hour. How many hours after the first starts are the travellers together?

9. If a person saves $120 each year, and puts the sum at simple interest at $3\frac{1}{2}\%$ at the end of each year, to how much will his property amount at the end of 18 years?

10. A ball is dropped from a window 32 feet above the pavement. Assuming the ball to be perfectly elastic and that on each rebound it rises to within $\frac{1}{6}$ of its former height, how far does it travel before coming to rest?

11. Two men travel from P to Q, leaving P at the same time. The distance from P to Q is 63 miles. The first travels 1 mile the first hour, 2 miles the second hour, 4 miles the third hour, and so on. The second travels 11 miles the first hour, $10\frac{4}{5}$ miles the second hour, $10\frac{3}{5}$ miles the third hour, and so on. Which is first to arrive at Q?

12. Find the geometric mean between .0729 and .0529.

13. Find the geometric mean between $\frac{.0144}{2.25}$ and $\frac{.0625}{576.}$.

14. Find the geometric mean between $\frac{x^2+xy}{xy-y^2}$ and $\frac{x^2-y^2}{xy}$.

15. Find the geometric mean between a^2-4a+4 and $4a^2+4a+1$.

16. The product of the first five terms of a G. P. is 243. Find the third term.

17. The digits of a number of three figures are in geometric progression. If units' and tens' digits are interchanged, the number formed exceeds the original number by 36. The sum of the digits is 14. Find the number.

18. A man travels $445\frac{1}{2}$ miles. He travels 10 miles the first day, and increases his speed one-half mile in each succeeding day. How many days does the journey require?

19. An A. P. has 19 terms such that the sum of the three middle terms is 3, and the sum of the first term and the last two terms is -13. Find the series.

20. Find the number of arithmetic means between 1 and 69, such that the ratio of the last mean to the first mean is 13.

21. Find an A. P. of 17 terms such that the sum of the first three terms is to the last term as 3 to 13, the first term being unity.

22. The sum of three successive terms of a geometric progression is 39 and the sum of their squares is 819. Find the series.

23. The sum of how many terms of the series 1, 3, 9 ⋯, is 3280?

24. Show that in any G. P., if each term is subtracted from the succeeding term, the differences form a G. P.

25. Find three numbers in A. P., such that the square of the first added to the product of the other two gives 16, and the square of the second added to the product of the other two gives 14.

PART II

X. INFINITE SERIES

246. Infinite Series (§ 178) may be developed by Division, or by Evolution.

Let it be required, for example, to divide 1 by $1 - x$.

$$\begin{array}{l}
1 - x)1(1 + x + x^2 + \cdots \\
\quad\;\; \underline{1 - x} \\
\qquad\quad x \\
\qquad\quad \underline{x - x^2} \\
\qquad\qquad\;\; x^2
\end{array}$$

Then, $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$ (1)

Again, let it be required to find the square root of $1 + x$.

$$\begin{array}{l|l}
 & 1 + x \;\big|\; 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \\
 & \underline{1} \\
2 + \frac{x}{2} & x \\
 & \underline{x + \frac{x^2}{4}} \\
2 + x - \frac{x^2}{8} & -\frac{x^2}{4}
\end{array}$$

Then, $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \cdots.$ (2)

It should be observed that the series, in (1) and (2), do not give the values of the first members for every value of x; thus, if x is a very large number, they evidently do not do so.

EXERCISE 62

Expand each of the following to four terms:

1. $\dfrac{5+4x}{1+3x}.$

2. $\dfrac{2+5x^2}{1+3x+x^2}.$

3. $\dfrac{1+3x}{1+7x-9x^2}.$

4. $\dfrac{5x}{5-x-3x^2}.$

5. $\sqrt{1+3x}$.

6. $\sqrt{1-5x^2}$.

7. $\sqrt{a^2+b^2}$.

8. $\sqrt[3]{a^3+b^3}$.

9. $\sqrt[3]{x^3+1}$.

10. $\sqrt{9a^2-16b^2}$.

CONVERGENCY AND DIVERGENCY OF SERIES

247. An infinite series is said to be *Convergent* when the sum of the first n terms approaches a fixed finite number as a limit (§ 125), when n is indefinitely increased.

An infinite series is said to be *Divergent* when the sum of the first n terms can be made numerically greater than any assigned number, however great, by taking n sufficiently great.

248. Consider, for example, the infinite series

$$1+x+x^2+x^3+\cdots.$$

I. Suppose $x=x_1$, where x_1 is numerically <1.

The sum of the first n terms is now

$$1+x_1+x_1^2+\cdots+x_1^{n-1}=\frac{1-x_1^n}{1-x_1} \text{ (§ 103, VII).}$$

If n be indefinitely increased, x_1^n decreases indefinitely in absolute value, and approaches the limit 0.

Then the fraction $\frac{1-x_1^n}{1-x_1}$ approaches the limit $\frac{1}{1-x_1}$.

That is, the sum of the first n terms approaches a fixed finite number as a limit, when n is indefinitely increased.

Hence, the series is *convergent* when x is numerically <1.

II. Suppose $x=1$.

In this case, each term of the series is equal to 1, and the sum of the first n terms is equal to n; and this sum can be made to exceed any assigned number, however great, by taking n sufficiently great.

Hence, the series is *divergent* when $x=1$.

III. Suppose $x = -1$.

In this case, the series takes the form $1 - 1 + 1 - 1 + \cdots$, and the sum of the first n terms is either 1 or 0 according as n is odd or even.

Hence, the series is neither convergent nor divergent when $x = -1$.

An infinite series which is neither convergent nor divergent is called an *Oscillating Series.*

IV. Suppose $x = x_1$, where x is numerically > 1.

The sum of the first n terms is now

$$1 + x_1 + x_1^2 + \cdots + x_1^{n-1} = \frac{x_1^n - 1}{x_1 - 1} \text{ (§ 103, VII).}$$

By taking n sufficiently great, $\frac{x_1^n - 1}{x_1 - 1}$ can be made to numerically exceed any assigned number, however great.

Hence, the series is *divergent* when x is numerically > 1.

249. Consider the infinite series

$$1 + x + x^2 + x^3 + \cdots,$$

developed by the fraction $\frac{1}{1 - x}$ (§ 246).

Let $x = .1$, in which case the series is convergent (§ 248).

The series now takes the form $1 + .1 + .01 + .001 + \cdots$, while the value of the fraction is $\frac{1}{.9}$ or $\frac{10}{9}$.

In this case, however great the number of terms taken, their sum will never exactly equal $\frac{10}{9}$.

But the sum approaches this value as a limit; for the series is a decreasing geometric progression, whose first term is 1, and ratio .1; and, by § 242, its sum to infinity is $\frac{1}{1 - .1}$, or $\frac{10}{9}$.

Thus, if an infinite series is *convergent*, the greater the number of terms taken, the more nearly does their sum approach the value of the expression from which the series was developed.

Again, let $x = 10$, in which case the series is divergent.

The series now takes the form $1 + 10 + 100 + 1000 + \cdots$, while the value of the fraction is $\frac{1}{1-10}$, or $-\frac{1}{9}$.

In this case the greater the number of terms taken, the more does their sum diverge from the value $-\frac{1}{9}$.

Thus, if an infinite series is *divergent*, the greater the number of terms taken, the more does their sum diverge from the value of the expression from which the series was developed.

It follows from the above that *an infinite series cannot be used for the purposes of demonstration if it is divergent.*

SUMMATION OF SERIES

250. The **Summation** of an infinite literal series is the process of finding an expression from which the series may be developed.

RECURRING SERIES

251. Consider the infinite series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots.$$

Here $$(3x^2) - 2x(2x) + x^2(1) = 0,$$

$$(4x^3) - 2x(3x^2) + x^2(2x) = 0, \text{ etc.}$$

That is, any three consecutive terms, as, for example, $2x$, $3x^2$, and $4x^3$, are so related that the third, minus $2x$ times the second, plus x^2 times the first, equals 0.

252. A **Recurring Series** is an infinite series of the form

$$a_0 + a_1x + a_2x^2 + \cdots,$$

where any $r + 1$ consecutive terms, as for example

$$a_nx^n,\ a_{n-1}x^{n-1},\ a_{n-2}x^{n-2},\ \cdots,\ a_{n-r}x^{n-r},$$

are so related that

$$a_n x^n + px(a_{n-1}x^{n-1}) + qx^2(a_{n-2}x^{n-2}) + \cdots + sx^r(a_{n-r}x^{n-r}) = 0;$$

$p, q, \cdots, s$ being constants.

The above recurring series is said to be of the *rth order*, and the expression

$$1 + px + px^2 + \cdots + sx^r$$

is called its *scale of relation.*

The recurring series of § 251 is of the second order, and its scale of relation is $1 - 2x + x^2$.

An infinite geometric series is a recurring series of the first order.

Thus, in the infinite geometric series

$$1 + x + x^2 + x^3 + \cdots,$$

any two consecutive terms, as for example x^3 and x^2, are so related that $(x^3) - x(x^2) = 0$; and the scale of relation is $1 - x$.

253. *To find the scale of relation of a recurring series.*

If the series is of the first order, the scale of relation may be found by dividing any term by the preceding term, and subtracting the result from 1.

If it is of the second order, $a_0, a_1, a_2, a_3, \cdots$, its consecutive coefficients, and $1 + px + qx^2$ its scale of relation, we shall have

$$\begin{cases} a_2 + pa_1 + qa_0 = 0, \\ a_3 + pa_2 + qa_1 = 0; \end{cases} \qquad (1)$$

from which p and q may be determined.

If the series is of the third order, $a_0, a_1, a_2, a_3, a_4, a_5, \cdots$, its consecutive coefficients, and $1 + px + qx^2 + rx^3$ its scale of relation, we shall have

$$\begin{cases} a_3 + pa_2 + qa_1 + ra_0 = 0, \\ a_4 + pa_3 + qa_2 + ra_1 = 0, \\ a_5 + pa_4 + qa_3 + ra_2 = 0; \end{cases}$$

from which p, q, and r may be determined.

To ascertain the order of a series, we may first make trial of a scale of relation of three terms.

If the result does not agree with the series, try a scale of four terms, five terms, and so on until the correct scale of relation is found.

If the series is assumed to be of too high an order, the equations corresponding to the assumed scale will not be independent (§ 43).

254. *To find the sum (§ 250) of a recurring series when its scale of relation is known.*

Let $1 + px + qx^2$ be the scale of relation of the series

$$a_0 + a_1x + a_2x^2 + \cdots.$$

Denoting the sum of the first n terms by S_n, we have

$$S_n = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}.$$

Then, $$pxS_n = pa_0x + pa_1x^2 + \cdots + pa_{n-2}x^{n-1} + pa_{n-1}x^n,$$

and $$qx^2S_n = qa_0x^2 + \cdots + qa_{n-3}x^{n-1} + qa_{n-2}x^n + qa_{n-1}x^{n+1}.$$

Adding these equations, and remembering that, by virtue of the scale of relation,

$$a_2 + pa_1 + qa_0 = 0, \cdots, a_{n-1} + pa_{n-2} + qa_{n-3} = 0,$$

the coefficients of x^2, x^3, $\cdots$, x^{n-1} become 0, and we have

$$S_n(1 + px + qx^2) = a_0 + (a_1 + pa_0)x + (pa_{n-1} + qa_{n-2})x^n + qa_{n-1}x^{n+1}.$$

Whence,

$$S_n = \frac{a_0 + (a_1 + pa_0)x + (pa_{n-1} + qa_{n-2})x^n + qa_{n-1}x^{n+1}}{1 + px + qx^2}; \quad (1)$$

which is a formula for the sum of the first n terms of a recurring series of the second order.

If x is so taken that the given series is convergent, x^n and x^{n+1} approach the limit 0, when n is indefinitely increased, and the fraction (1) approaches the limit

$$\frac{a_0 + (a_1 + pa_0)x}{1 + px + qx^2}.$$

If this fraction be expanded into an infinite series by division, we obtain the given series; but it is only when the series is convergent that it expresses the value of the fraction.

Then, the sum of the given series (§ 250) is given by the formula

$$S = \frac{a_0 + (a_1 + pa_0)x}{1 + px + qx^2}. \tag{2}$$

If $q = 0$, the series is of the first order, and $a_1 + pa_0 = 0$; then

$$S = \frac{a_0}{1 + px}; \tag{3}$$

which is a formula for the sum of a recurring series of the first order. (Compare § 242.)

In like manner, we shall find the formula

$$S = \frac{a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2}{1 + px + qx^2 + rx^3} \tag{4}$$

for the sum of a recurring series of the third order.

255. A recurring series is formed by the expansion in an infinite series of a fraction, called the *generating fraction.*

The operation of summation reproduces the fraction, the process being the reverse of that of § 268.

256. *Ex.* Find the sum of the series

$$2 + x + 5x^2 + 7x^3 + 17x^4 + \cdots.$$

To determine the scale of relation, we first assume the series to be of the second order (§ 253).

Substituting $a_0 = 2$, $a_1 = 1$, $a_2 = 5$, $a_3 = 7$, in (1), § 253,

$$\begin{cases} 5 + p + 2q = 0, \\ 7 + 5p + q = 0. \end{cases}$$

Solving these equations, $p = -1$, $q = -2$.

To ascertain if $1 - x - 2x^2$ is the correct scale of relation, consider the fifth term.

Since $17x^4 + (-x)(7x^3) + (-2x^2)(5x^2)$ is equal to 0, it follows that $1 - x - 2x^2$ is the correct scale.

Substituting the values of a_0, a_1, p, and q in (2),

$$S = \frac{2 + (1 - 2)x}{1 - x - 2x^2} = \frac{2 - x}{1 - x - 2x^2}.$$

The result may be verified by expansion.

EXERCISE 63

Find the sum of the following:

1. $4 + x + 7x^2 - 5x^3 + 19x^4 + \cdots$.
2. $1 - 13x - 23x^2 - 85x^3 - 239x^4 + \cdots$.
3. $1 + 5x + 21x^2 + 85x^3 + 341x^4 + \cdots$.
4. $5 - 13x + 35x^2 - 97x^3 + 275x^4 + \cdots$.
5. $3 + 10x + 36x^2 + 136x^3 + 528x^4 + \cdots$.
6. $3 + x + 33x^2 + 109x^3 + 657x^4 + \cdots$.
7. $1 + 2x - 3x^2 + 6x^3 - 7x^4 + 10x^5 - 11x^6 + \cdots$.
8. $1 - 2x - x^2 - 7x^3 - 18x^4 - 59x^5 - 181x^6 + \cdots$.

THE DIFFERENTIAL METHOD

257. If the first term of a series be subtracted from the second, the second from the third, and so on, the series formed is called the *first order of differences* of the given series.

The first order of differences of this new series is called the *second order of differences* of the given series; and so on.

Thus, in the series

$$1, \quad 8, \quad 27, \quad 64, \quad 125, \quad 216, \quad \cdots,$$

the successive orders of differences are as follows:

1st order,	7, 19, 37, 61, 91, ...
2d order,	12, 18, 24, 30, ...
3d order,	6, 6, 6, ...
4th order,	0, 0, ...

The **Differential Method** is a method for finding any term, or the sum of any number of terms of a series, by means of its successive orders of differences.

258. *To find any term of the series*

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \cdots, \quad a_n, \quad a_{n+1}, \quad \cdots.$$

The successive orders of differences are as follows:

1st order, $a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,\ \cdots,\ a_{n+1} - a_n,\ \cdots$.
2d order, $a_3 - 2a_2 + a_1,\ a_4 - 2a_3 + a_2,\ \cdots$.
3d order, $a_4 - 3a_3 + 3a_2 - a_1,\ \cdots$; etc.

Denoting the first terms of the 1st, 2d, 3d, ⋯, orders of differences by d_1, d_2, d_3, ⋯, respectively, we have

$d_1 = a_2 - a_1$; whence, $a_2 = a_1 + d_1$.
$d_2 = a_3 - 2a_2 + a_1$; whence,
$a_3 = -a_1 + 2a_2 + d_2 = -a_1 + 2a_1 + 2d_1 + d_2 = a_1 + 2d_1 + d_2$.
$d_3 = a_4 - 3a_3 + 3a_2 - a_1$; whence,
$a_4 = a_1 - 3a_2 + 3a_3 + d_3 = a_1 + 3d_1 + 3d_2 + d_3$; etc.

It will be observed, in the values of a_2, a_3, and a_4, that the coefficients of the terms are the same as the coefficients of the terms in the expansion by the Binomial Theorem of $a + x$ to the *first*, *second*, and *third* powers, respectively.

We will now prove by Mathematical Induction that this law holds for any term of the given series.

Assume the law to hold for the nth term, a_n; then the coefficients of the terms will be the same as the coefficients of the terms in the expansion by the Binomial Theorem of $a + x$ to the $(n-1)$th power; that is,

$$a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{\underline{|2}}d_2 + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}d_3 + \cdots. \quad (1)$$

If the law holds for the nth term of any series, it must also hold for the nth term of the first order of differences; or,

$$a_{n+1} - a_n = d_1 + (n-1)d_2 + \frac{(n-1)(n-2)}{\underline{|2}}d_3 + \cdots. \quad (2)$$

Adding (1) and (2), we have

$$a_{n+1} = a_1 + [(n-1)+1]d_1 + \frac{n-1}{\underline{|2}}[(n-2)+2]d_2 + \frac{(n-1)(n-2)}{\underline{|3}}[(n-3)+3]d_3 + \cdots$$

$$= a_1 + nd_1 + \frac{n(n-1)}{\underline{|2}}d_2 + \frac{n(n-1)(n-2)}{\underline{|3}}d_3 + \cdots. \quad (3)$$

This result is in accordance with the above law.

Hence, if the law holds for the nth term of the given series, it holds for the $(n+1)$th term; but we know that it holds for the fourth term, and hence it holds for the fifth term; and so on.

Therefore, (1) holds for any term of the given series.

If the differences finally become zero, the value of a_n can be obtained exactly.

259. *To find the sum of the first n terms of the series*

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \cdots. \tag{1}$$

Let S denote the sum of the first n terms.

Then S is the $(n+1)$th term of the series

$$0,\ a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \cdots. \tag{2}$$

The first order of differences of (2) is the same as (1); whence, the 2d order of differences of (2) is the same as the 1st order of differences of (1), the 3d order of (2) is the same as the 2d order of (1), and so on.

Then, if d_1, d_2, $\cdots$, represent the first terms of the 1st, 2d, $\cdots$, orders of differences of (1), a_1, d_1, d_2, $\cdots$, will be the first terms of the 1st, 2d, 3d, $\cdots$, orders of differences of (2).

Putting $a_1 = 0$, $d_1 = a_1$, $d_2 = d_1$, etc., in (3), § 258,

$$S = na_1 + \frac{n(n-1)}{\underline{|2}}d_1 + \frac{n(n-1)(n-2)}{\underline{|3}}d_2 + \cdots. \tag{3}$$

260. *Ex.* Find the twelfth term, and the sum of the first twelve terms, of the series 1, 8, 27, 64, 125, $\cdots$.

Here, $n = 12$, $a_1 = 1$.

Also, $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, and $d_4 = 0$ (§ 257).

Substituting in (1), § 258, the twelfth term

$$= 1 + 11 \cdot 7 + \frac{11 \cdot 10}{1 \cdot 2} \cdot 12 + \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} \cdot 6 = 1728.$$

Substituting in (3), § 259, the sum of the first twelve terms

$$= 12 + \frac{12 \cdot 11}{1 \cdot 2} \cdot 7 + \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} \cdot 12 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 6 = 6084.$$

261. Piles of Shot.

Ex. If shot be piled in the shape of a pyramid with a triangular base, each side of which exhibits 9 shot, find the number in the pile.

The number of shot in the first five courses are 1, 3, 6, 10, and 15, respectively; we have then to find the sum of the first nine terms of the series 1, 3, 6, 10, 15, ⋯; the successive orders of differences are as follows:

1st order,	.	2, 3, 4, 5, ⋯.
2d order,	.	1, 1, 1, ⋯.
3d order,	.	0, 0, ⋯.

Putting $n = 9$, $a_1 = 1$, $d_1 = 2$, $d_2 = 1$ in (3), § 259,

$$S = 9 + \frac{9 \cdot 8}{1 \cdot 2} \cdot 2 + \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot 1 = 165.$$

EXERCISE 64

1. Find the first term of the sixth order of differences of the series 3, 5, 11, 27, 67, 159, 375, ⋯.

2. Find the 15th term, and the sum of the first 15 terms, of the series 1, 9, 21, 37, 57, ⋯.

3. Find the 14th term, and the sum of the first 14 terms, of the series 5, 14, 15, 8, -7, ⋯.

4. Find the sum of the first n multiples of 3.

5. If shot be piled in the shape of a pyramid with a square base, each side of which exhibits 25 shot, find the number in the pile.

6. Find the 13th term, and the sum of the first 13 terms, of the series 1, 3, 9, 25, 57, 111, ⋯.

7. Find the 10th term, and the sum of the first 10 terms, of the series 4, -2, 10, 4, -56, -206, ⋯.

8. Find the sum of the squares of the first n multiples of 2.

9. Find the nth term, and the sum of the first n terms, of the series 1, -3, -13, -17, -3, 41, ⋯.

10. Find the number of shot in a pile of 9 courses, with a rectangular base, if the number of shot in the longest side of the base is 24.

11. Find the number of shot in a truncated pile of 8 courses, with a rectangular base, if the number of shot in the length and breadth of the base are 20 and 14, respectively.

12. Find the 12th term, and the sum of the first 12 terms, of the series 1, 13, 49, 139, 333, 701, 1333, $\cdots$.

13. Find the 9th term, and the sum of the first 9 terms, of the series 20, 4, -36, -132, -356, -820, -1676, $\cdots$.

14. Find the sum of the fourth powers of the first n natural numbers.

15. Find the number of shot in a pile with a rectangular base, if the number of shot in the length and breadth of the base are m and n, respectively.

16. How many shot are contained in a truncated pile of n courses, whose bases are triangles, if the number of shot in each side of the upper base is m?

INTERPOLATION

262. Interpolation is the process of introducing between the terms of a series other terms conforming to the law of the series.

Its usual application is in finding intermediate numbers between those given in Mathematical Tables.

The operation is effected by giving *fractional* values to n in (1), § 258.

The method of Interpolation rests on the assumption that a formula which has been proved for an integral value of n, holds also when n is fractional.

263. *Ex.* Given $\sqrt{5} = 2.2361$, $\sqrt{6} = 2.4495$, $\sqrt{7} = 2.6458$, $\sqrt{8} = 2.8284$, $\cdots$; find $\sqrt{6.3}$.

In this case the successive orders of differences are:

$$\begin{array}{llll} .2134, & .1963, & .1826, & \cdots. \\ & -.0171, & -.0137, & \cdots. \\ & & .0034, & \cdots. \end{array}$$

Whence, $d_1 = .2134$, $d_2 = -.0171$, $d_3 = .0034$, $\cdots$.

Now, the required term is distant 1.3 intervals from $\sqrt{5}$.

Substituting $n = 2.3$ in (1), § 258, we have, approximately,

$$\sqrt{6.3} = 2.2361 + 1.3 \times .2134 + \frac{1.3 \times .3}{1 \times 2}(-.0171)$$

$$+ \frac{1.3 \times .3 \times -.7}{1 \times 2 \times 3} \times .0034$$

$$= 2.2361 + .2774 - .0033 - .0002 = 2.5100.$$

EXERCISE 65

1. Given $\log 26 = 1.4150$, $\log 27 = 1.4314$, $\log 28 = 1.4472$, $\log 29 = 1.4624$, $\cdots$; find $\log 26.7$.

2. Given $\sqrt[3]{91} = 4.49794$, $\sqrt[3]{92} = 4.51436$, $\sqrt[3]{93} = 4.53066$, $\sqrt[3]{94} = 4.54684$, $\cdots$; find $\sqrt[3]{92.5}$.

3. The reciprocal of 35 is .02857; of 36, .02778; of 37, .02703; of 38, .02632; etc. Find the reciprocal of 36.28.

4. Given $\log 124 = 2.09342$, $\log 125 = 2.09691$, $\log 126 = 2.10037$, $\log 127 = 2.10380$, $\cdots$; find $\log 125.36$.

5. Given $21^3 = 9261$, $22^3 = 10648$, $23^3 = 12167$, $24^3 = 13824$, and $25^3 = 15625$; find the cube of $21\frac{1}{2}$.

6. Given $\log 61 = 1.78533$, $\log 62 = 1.79239$, $\log 63 = 1.79934$, $\log 64 = 1.80618$, $\cdots$; find $\log 63.527$.

XI. UNDETERMINED COEFFICIENTS

THE THEOREM OF UNDETERMINED COEFFICIENTS

264. An important method for expanding expressions into series is based on the following theorem:

If the series $A + Bx + Cx^2 + Dx^3 + \cdots$ is always equal to the series $A' + B'x + C'x^2 + D'x^3 + \cdots$, when x has any value which makes both series convergent, the coefficients of like powers of x in the series will be equal; that is, $A = A'$, $B = B'$, $C = C'$.

265. Before giving the proof of the Theorem of Undetermined Coefficients, we will prove two theorems in regard to infinite series.

First, if the infinite series

$$a + bx + cx^2 + dx^2 + \cdots$$

is convergent for some finite value of x, it is *finite* for this value of x (§ 247), and therefore finite when $x = 0$.

Hence, the series is convergent when $x = 0$.

Second, if the infinite series

$$ax + bx^2 + cx^3 + \cdots$$

is convergent for some finite value of x, it equals 0 when $x = 0$.

For, $ax + bx^2 + cx^3 + \cdots$ is finite for this value of x, and hence $a + bx + cx^2 + \cdots$ is finite for this value of x.

Then, $a + bx + cx^2 + \cdots$ is finite when $x = 0$; and therefore $x(a + bx + cx^2 + \cdots)$, or $ax + bx^2 + cx^3 + \cdots$, equals 0 when $x = 0$.

266. Proof of the Theorem of Undetermined Coefficients.

The equation

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots \quad (1)$$

is satisfied when x has any value which makes both members convergent; and since both members are convergent when $x = 0$ (§ 265), the equation is satisfied when $x = 0$.

Putting $x=0$, we have, by § 265,

$$Bx + Cx^2 + Dx^3 + \cdots = 0, \text{ and } B'x + C'x^2 + D'x^3 + \cdots = 0.$$

Whence, $A = A'$.

Subtracting A from the first member of (1), and its equal A' from the second member, we have

$$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots.$$

Dividing each term by x,

$$B + Cx + Dx^2 + \cdots = B' + C'x + D'x^2 + \cdots. \qquad (2)$$

The members of this equation are finite for the same values of x as the given series (§ 265).

Then, they are convergent, and therefore *equal*, for the same values of x as the given series.

Then the equation (2) is satisfied when $x=0$.

Putting $x=0$, we have $B = B'$.

Proceeding in this way, we may prove $C = C'$, etc.

267. The theorem of § 264 holds when either or both of the given series are finite.

EXPANSION OF FRACTIONS

268. I. Expand $\dfrac{2-3x^2-x^3}{1-2x+3x^2}$ in ascending powers of x.

Assume $$\frac{2-3x^2-x^3}{1-2x+3x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots, \qquad (1)$$

where $A, B, C, D, E, \cdots$, are numbers independent of x.

Clearing of fractions, and collecting the terms in the second member involving like powers of x, we have

$$2-3x^2-x^3 = A + \begin{array}{r|}B\\-2A\\ \end{array}\,x + \begin{array}{r|}C\\-2B\\+3A\end{array}\,x^2 + \begin{array}{r|}D\\-2C\\+3B\end{array}\,x^3 + \begin{array}{r|}E\\-2D\\+3C\end{array}\,x^4 + \cdots. \qquad (2)$$

A vertical line, called a *bar*, is often used in place of parentheses.

Thus, $\begin{array}{r|}+\ B\\-2A\end{array}\,x$ is equivalent to $(B-2A)x$.

The second member of (1) must express the value of the fraction for every value of x which makes the series convergent (§ 249); and therefore equation (2) is satisfied when x has any value which makes the second member convergent.

Then, by § 267, the coefficients of like powers of x in (2) must be equal; that is,

$$\begin{aligned}
A &= 2. \\
B - 2A &= 0;\ \text{or, } B = 2A = 4. \\
C - 2B + 3A &= -3;\ \text{or, } C = 2B - 3A - 3 = -1. \\
D - 2C + 3B &= -1;\ \text{or, } D = 2C - 3B - 1 = -15. \\
E - 2D + 3C &= 0;\ \text{or, } E = 2D - 3C = -27;\ \text{etc.}
\end{aligned}$$

Substituting these values in (1), we have

$$\frac{2 - 3x^2 - x^3}{1 - 2x + 3x^2} = 2 + 4x - x^2 - 15x^3 - 27x^4 - \cdots.$$

The result may be verified by division.

The series expresses the value of the fraction only for such values of x as make it convergent (§ 249).

If the numerator and denominator contain only *even* powers of x, the operation may be abridged by assuming a series containing only the even powers of x.

Thus, if the fraction were $\frac{2 + 4x^2 - x^4}{1 - 3x^2 + 5x^4}$, we should assume it equal to $A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \cdots$.

In like manner, if the numerator contains only *odd* powers of x, and the denominator only even powers, we should assume a series containing only the odd powers of x.

If every term of the numerator contains x, we may assume a series commencing with the lowest power of x in the numerator.

If every term of the denominator contains x, we determine by actual division what power of x will occur in the first term of the expansion, and then assume the fraction equal to a series commencing with this power of x, the exponents of x in the succeeding terms increasing by unity as before.

2. Expand $\frac{1}{3x^2 - x^3}$ in ascending powers of x.

Dividing 1 by $3x^2$, the quotient is $\frac{x^{-2}}{3}$; we then assume,

$$\frac{1}{3x^2 - x^3} = Ax^{-2} + Bx^{-1} + C + Dx + Ex^2 + \cdots. \qquad (3)$$

Clearing of fractions,

$$1 = 3A + \begin{array}{c|} 3B \\ -A \end{array}\, x + \begin{array}{c|} 3C \\ -B \end{array}\, x^2 + \begin{array}{c|} 3D \\ -C \end{array}\, x^3 + \begin{array}{c|} 3E \\ -D \end{array}\, x^4 + \cdots.$$

Equating coefficients of like powers of x,

$3A = 1$, $3B - A = 0$, $3C - B = 0$, $3D - C = 0$, $3E - D = 0$; etc.

Whence, $A = \frac{1}{3}$, $B = \frac{1}{9}$, $C = \frac{1}{27}$, $D = \frac{1}{81}$, $E = \frac{1}{243}$, etc.

Substituting in (3), $\frac{1}{3x^2 - x^3} = \frac{x^{-2}}{3} + \frac{x^{-1}}{9} + \frac{1}{27} + \frac{x}{81} + \frac{x^2}{243} + \cdots.$

In Ex. 1, $E = 2D - 3C$; that is, the coefficient of x^4 equals twice the coefficient of the preceding term, minus three times the coefficient of the next but one preceding.

It is evident that this law holds for the succeeding terms; thus, the coefficient of x^5 is $2 \times (-27) - 3 \times (-15)$, or -9.

After the law of coefficients has been found in any expansion, the terms may be found more easily than by long division; and for this reason the method of § 268 is to be preferred when a large number of terms is required.

The law for Ex. 2 is that each coefficient is one-third the preceding.

EXERCISE 66

Expand each of the following to five terms in ascending powers of x:

1. $\frac{4 + 2x}{1 - 2x}$.
2. $\frac{1 - 8x}{1 + 5x}$.
3. $\frac{2 + x^2}{1 - 2x^2}$.
4. $\frac{5x}{1 + 6x^2}$.
5. $\frac{3 + x + x^2}{5 + 2x - 7x^2}$.
6. $\frac{4 + x - 3x^2}{1 - 2x^2}$.
7. $\frac{x - 5x^3 + 8x^4}{2 - 3x + x^2}$.
8. $\frac{4x^2 - 3x^3}{6 - 4x - 5x^3}$.
9. $\frac{1 + 2x + 4x^2}{1 - 2x + 3x^2}$.
10. $\frac{5 + 6x^2}{2 - 3x^2 + x^3}$.
11. $\frac{1 - 6x^2 + 4x^3}{x + x^2 - 2x^3}$.
12. $\frac{1 - 2x + 3x^3}{2x^3 + 4x^5 + x^6}$.

EXPANSION OF SURDS

269. *Ex.* Expand $\sqrt{1-x}$ in ascending powers of x.

Assume $\sqrt{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots.$ (1)

Squaring both members, we have, by § 167,

$$1 - x = A^2 \left|\begin{matrix} x + \\ +2AB \end{matrix}\right. \left|\begin{matrix} B^2 \\ +2AC \end{matrix}\right| x^2 \left.\begin{matrix} \\ +2AD \\ +2BC \end{matrix}\right| x^3 + \left.\begin{matrix} C^2 \\ +2AE \\ +2BD \end{matrix}\right| x^4 + \cdots.$$

Equating coefficients of like powers of x,

$$A^2 = 1;\ \text{or, } A = 1.$$

$$2AB = -1;\ \text{or, } B = -\frac{1}{2A} = -\frac{1}{2}.$$

$$B^2 + 2AC = 0;\ \text{or, } C = -\frac{B^2}{2A} = -\frac{1}{8}.$$

$$2AD + 2BC = 0;\ \text{or, } D = -\frac{BC}{A} = -\frac{1}{16}.$$

$$C^2 + 2AE + 2BD = 0;\ \text{or, } E = -\frac{C^2 + 2BD}{2A} = -\frac{5}{128};\ \text{etc.}$$

Substituting these values in (1), we have

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots.$$

The result may be verified by Evolution.

The series expresses the value of $\sqrt{1-x}$ only for such values of x as make it convergent.

EXERCISE 67

Expand each of the following to five terms in ascending powers of x:

1. $\sqrt{1+2x}$.
2. $\sqrt{1-3x}$.
3. $\sqrt{1-4x+x^2}$.
4. $\sqrt{1+x-x^2}$.
5. $\sqrt[3]{1+6x}$.
6. $\sqrt[3]{1-x-2x^2}$.

PARTIAL FRACTIONS

270. If the denominator of a fraction can be resolved into factors, each of the first degree in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions*, whose denominators are factors of the given denominator, and whose numerators are independent of x.

271. CASE I. *No factors of the denominator equal.*

1. Separate $\frac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

Assume $$\frac{19x+1}{(3x-1)(5x+2)}=\frac{A}{3x-1}+\frac{B}{5x+2}, \qquad (1)$$

where A and B are numbers independent of x.

Clearing of fractions, $19x+1=A(5x+2)+B(3x-1)$.

Or, $$19x+1=(5A+3B)x+2A-B. \qquad (2)$$

The second member of (1) must express the value of the given fraction for every value of x.

Hence, equation (2) is satisfied by every value of x; and by § 267, the coefficients of like powers of x in the two members are equal.

That is, $$5A+3B=19,$$

and $$2A-B=1.$$

Solving these equations, we obtain $A=2$ and $B=3$.

Substituting in (1), $$\frac{19x+1}{(3x-1)(5x+2)}=\frac{2}{3x-1}+\frac{3}{5x+2}.$$

The result may be verified by finding the sum of the partial fractions.

2. Separate $\frac{x+4}{2x-x^2-x^3}$ into partial fractions.

The factors of $2x-x^2-x^3$ are x, $1-x$, and $2+x$ (§ 103, III, VIII).

Assume then, $$\frac{x+4}{2x-x^2-x^3}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{2+x}.$$

Clearing of fractions, we have

$$x+4=A(1-x)(2+x)+Bx(2+x)+Cx(1-x).$$

This equation, being satisfied by every value of x, is satisfied when $x=0$.

Putting $x = 0$, we have $4 = 2A$, or $A = 2$.

Again, the equation is satisfied when $x = 1$.

Putting $x = 1$, we have $5 = 3B$, or $B = \frac{5}{3}$.

The equation is also satisfied when $x = -2$.

Putting $x = -2$, we have $2 = -6C$, or $C = -\frac{1}{3}$.

Then, $\dfrac{x+4}{2x - x^2 - x^3} = \dfrac{2}{x} + \dfrac{\frac{5}{3}}{1-x} + \dfrac{-\frac{1}{3}}{2+x} = \dfrac{2}{x} + \dfrac{5}{3(1-x)} - \dfrac{1}{3(2+x)}$.

To find the value of A, in Ex. 2, we give to x *such a value as will make the coefficients of B and C equal to zero;* and we proceed in a similar manner to find the values of B and C.

This method of finding A, B, and C is usually shorter than that used in Ex. 1.

Case II. *All the factors of the denominator equal.*

Let it be required to separate $\dfrac{x^2 - 11x + 26}{(x-3)^3}$ into partial fractions.

Substituting $y + 3$ for x, the fraction becomes

$$\frac{(y+3)^2 - 11(y+3) + 26}{y^3} = \frac{y^2 - 5y + 2}{y^3} = \frac{1}{y} - \frac{5}{y^2} + \frac{2}{y^3}.$$

Replacing y by $x - 3$, the result takes the form

$$\frac{1}{x-3} - \frac{5}{(x-3)^2} + \frac{2}{(x-3)^3}.$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x - 3$ beginning with the first and ending with the third.

Similar considerations hold with respect to any example under Case II; the number of partial fractions in any case being the same as the number of equal factors in the denominator of the given fraction.

Ex. Separate $\dfrac{6x+5}{(3x+5)^2}$ into partial fractions.

In accordance with the above principle, we assume the given fraction equal to the sum of *two* partial fractions, whose denominators are the powers of $3x+5$ beginning with the first and ending with the *second*.

That is,
$$\frac{6x+5}{(3x+5)^2}=\frac{A}{3x+5}+\frac{B}{(3x+5)^2}.$$

Clearing of fractions, $6x+5=A(3x+5)+B.$
$$=3Ax+5A+B.$$

Equating coefficients of like powers of x,
$$3A=6,$$
and
$$5A+B=5.$$

Solving these equations, $A=2$ and $B=-5.$

Whence,
$$\frac{6x+5}{(3x+5)^2}=\frac{2}{3x+5}-\frac{5}{(3x+5)^2}.$$

Case III. *Some of the factors of the denominator equal.*

Ex. Separate $\frac{x^2-4x+3}{x(x+1)^2}$ into partial fractions.

The method in Case III is a combination of the methods of Cases I and II; we assume,
$$\frac{x^2-4x+3}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

Clearing of fractions,
$$x^2-4x+3=A(x+1)^2+Bx(x+1)+Cx$$
$$=(A+B)x^2+(2A+B+C)x+A.$$

Equating coefficients of like powers of x,
$$A+B=1,$$
$$2A+B+C=-4,$$
and
$$A=3.$$

Solving these equations, $A=3$, $B=-2$, and $C=-8$.

Whence,
$$\frac{x^2-4x+3}{x(x+1)^2}=\frac{3}{x}-\frac{2}{x+1}-\frac{8}{(x+1)^2}.$$

The following general rule for Case III will be found convenient:

A fraction of the form $\frac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be assumed equal to
$$\frac{A}{x+a}+\frac{B}{x+b}+\cdots+\frac{E}{x+m}+\frac{F}{(x+m)^2}+\cdots+\frac{K}{(x+m)^r}+\cdots,$$

single factors like $x+a$ and $x+b$ having single partial fractions corresponding, arranged as in Case I; and repeated factors like $(x+m)$ having r partial fractions corresponding, arranged as in Case II.

272. If the degree of the numerator is equal to, or greater than, that of the denominator, the preceding methods are inapplicable.

In such a case, we divide the numerator by the denominator until a remainder is obtained which is of a lower degree than the denominator.

Ex. Separate $\frac{x^3-3x^2-1}{x^2-x}$ into an integral expression and partial fractions.

Dividing x^3-3x^2-1 by x^2-x, the quotient is $x-2$, and the remainder $-2x-1$; we then have

$$\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{-2x-1}{x^2-x}. \qquad (1)$$

We can now separate $\frac{-2x-1}{x^2-x}$ into partial fractions by the method of Case I; the result is $\frac{1}{x}-\frac{3}{x-1}$.

Substituting in (1), $\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{1}{x}-\frac{3}{x-1}$.

Another way to solve the above example is to combine the methods of §§ 268 and 271, and assume the given fraction equal to

$$Ax+B+\frac{C}{x}+\frac{D}{x-1}.$$

273. If the denominator of a fraction can be resolved into factors partly of the first and partly of the second, or all of the second degree, in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more partial fractions, whose denominators are factors of the given denominator, and whose numerators are independent of x in the case of fractions corresponding to factors of the first degree, and of the form $Ax+B$ in the case of fractions corresponding to factors of the second degree.

The only exceptions occur when the factors of the denominator are of the second degree and all equal.

Ex. Separate $\dfrac{1}{x^3+1}$ into partial fractions.

The factors of the denominator are $x+1$ and x^2-x+1.

Assume then $$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}. \qquad (1)$$

Clearing of fractions, $1=A(x^2-x+1)+(Bx+C)(x+1)$.

Or, $1=(A+B)x^2+(-A+B+C)x+A+C$.

Equating coefficients of like powers of x,

$$A+B=0,$$
$$-A+B+C=0,$$
and $$A+C=1.$$

Solving these equations, $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$.

Substituting in (1), $$\frac{1}{x^3+1}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}.$$

EXERCISE 68

Separate into partial fractions:

1. $\dfrac{-1}{x^2-9x+20}$.
2. $\dfrac{15x-27}{10x^2+x-21}$.
3. $\dfrac{2x+3}{x^2-x-12}$.
4. $\dfrac{12x+18}{x^3+3x^2-18x}$.
5. $\dfrac{43x-31}{30x^2-12x-306}$.
6. $\dfrac{x^3+4x^2+2x+3}{(x^2+1)(x^2+x+1)}$.
7. $\dfrac{3x-7}{x^2-2x-8}$.
8. $\dfrac{6x^2-12}{x^4-5x^2+4}$.
9. $\dfrac{x^2-15x+3}{x^2-3x-28}$.
10. $\dfrac{5x^2+16x-2}{x^3+4x^2-3x-18}$.

11. $\dfrac{8x^4+16x^3-10x^2-28x+11}{2x^2+x-3}$.

12. $\dfrac{59x-53}{12x^2-25x+12}$.

13. $\dfrac{2x^2-11x+19}{x^3-6x^2+12x-8}$.

14. $\dfrac{2x^3-3x-8}{(x^2+x-2)^2}$.

15. $\dfrac{2x^3+x^2+5x}{(x^2+2x+1)(x^2-x+1)}$.

16. $\dfrac{x^4+2x^3-9x^2+7x}{x^4-4x^3+6x^2-4x+1}$.

17. $\dfrac{12x^4+19x^2-7x}{4x^4+1}$.

18. $\dfrac{x^3-x}{x(x^2+x+1)+2(x^2+x+1)}$.

19. $\dfrac{2x^2-2}{x^4+x^2+1}$.

20. $\dfrac{2x-3}{4x^3-x}$.

REVERSION OF SERIES

274. To *revert* a given series $y = a + bx^m + cx^n + \cdots$ is to express x as a series proceeding in ascending powers of y.

Ex. Revert the series $y = 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$.

Assume $$x = Ay + By^2 + Cy^3 + Dy^4 + \cdots. \quad (1)$$

Substituting in this the given value of y,

$$\begin{aligned} x = {} & A(2x - 3x^2 + 4x^3 - 5x^4 + \cdots) \\ & + B(4x^2 + 9x^4 - 12x^3 + 16x^4 + \cdots) \\ & + C(8x^3 - 36x^4 + \cdots) + D(16x^4 + \cdots) + \cdots. \end{aligned}$$

That is,
$$x = 2Ax \begin{array}{r|} -3A \\ +4B \\ \\ \\ \end{array} x^2 \begin{array}{r|} +4A \\ -12B \\ +8C \\ \\ \end{array} x^3 \begin{array}{r|} -5A \\ +25B \\ -36C \\ +16D \end{array} x^4 + \cdots.$$

Equating coefficients of like powers of x,

$$2A = 1;$$
$$-3A + 4B = 0;$$
$$4A - 12B + 8C = 0;$$
$$-5A + 25B - 36C + 16D = 0; \text{ etc.}$$

Solving, $A = \frac{1}{2},\ B = \frac{3}{8},\ C = \frac{5}{16},\ D = \frac{35}{128}$, etc.

Substituting in (1), $x = \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \frac{35}{128}y^4 + \cdots$.

If the even powers of x are wanting in the given series, the operation may be abridged by assuming x equal to a series containing only the *odd* powers of y.

EXERCISE 69

Revert each of the following to four terms:

1. $y = x + 3x^2 + 5x^3 + 7x^4 + \cdots$.
2. $y = x - 2x^2 + 3x^3 - 4x^4 + \cdots$.
3. $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$.
4. $y = 2x + 5x^2 + 8x^3 + 11x^4 + \cdots$.
5. $y = \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{6} - \frac{x^4}{8} + \cdots$.
6. $y = \frac{x}{\lfloor 2} + \frac{x^2}{\lfloor 3} + \frac{x^3}{\lfloor 4} + \frac{x^4}{\lfloor 5} + \cdots$.
7. $y = 2x - 4x^3 + 6x^5 - 8x^7 + \cdots$.
8. $y = \frac{x}{2} + \frac{x^3}{4} + \frac{x^5}{6} + \frac{x^7}{8} + \cdots$.

XII. PERMUTATIONS AND COMBINATIONS

275. The different orders in which things can be arranged are called their **Permutations.**

Thus, the permutations of the letters *a, b, c,* taken two at a time, are *ab, ac, ba, bc, ca, cb*; and their permutations, taken three at a time, are *abc, acb, bac, bca, cab, cba.*

276. The **Combinations** of things are the different collections which can be formed from them without regard to the order in which they are placed.

Thus, the combinations of the letters *a, b, c,* taken two at a time, are *ab, bc, ca*; for though *ab* and *ba* are different permutations, they form the same combination.

277. *To find the number of permutations of n different things taken two at a time.*

Consider the n letters, $a, b, c, \cdots$.

In making any particular permutation of two letters, the first letter may be any one of the n; that is, the first place can be filled in n different ways.

After the first place has been filled, the second place can be filled with any one of the remaining $n-1$ letters.

Then, the whole number of permutations of the letters taken two at a time is $n(n-1)$.

We will now consider the general case.

278. *To find the number of permutations of n different things taken r at a time.*

Consider the n letters $a, b, c, \cdots$.

In making any particular permutation of r letters, the first letter may be any one of the n.

After the first place has been filled, the second place can be filled with any one of the remaining $n-1$ letters.

After the second place has been filled, the third place can be filled in $n-2$ different ways.

Continuing in this way, the rth place can be filled in

$$n-(r-1), \text{ or } n-r+1 \text{ different ways.}$$

Then, the whole number of permutations of the letters taken r at a time is given by the formula

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1). \qquad (1)$$

The number of permutations of n different things taken r at a time is usually denoted by the symbol $_nP_r$.

279. If *all* the letters are taken, $r=n$, and (1) becomes

$$_nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = \underline{|n}. \qquad (2)$$

Hence, the number of permutations of n different things taken n at a time equals the product of the natural numbers from 1 to n inclusive. (See note, § 181.)

280. *To find the number of combinations of n different things taken r at a time.*

The number of *permutations* of n different things taken r at a time is $\quad n(n-1)(n-2)\cdots(n-r+1)$ (§ 278).

But, by § 279, each combination of r different things may have $\underline{|r}$ permutations.

Hence, the number of *combinations* of n different things taken r at a time equals the number of permutations divided by $\underline{|r}$.

That is,
$$_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}. \qquad (3)$$

The number of combinations of n different things taken r at a time is usually denoted by the symbol $_nC_r$.

281. Multiplying both terms of the fraction (3) by the product of the natural numbers from 1 to $n-r$ inclusive, we have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)\cdot(n-r)\cdots 2\cdot 1}{\underline{|r} \times 1\cdot 2\cdots(n-r)} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}};$$

which is another form of the result.

282. The number of combinations of n different things taken r at a time equals the number of combinations taken $n-r$ at a time.

For, for every selection of r things out of n, we leave a selection of $n-r$ things.

The theorem may also be proved by substituting $n-r$ for r, in the result of § 281.

283. Examples.

1. How many changes can be rung with 10 bells, taking 7 at a time?

Putting $n=10$, $r=7$, in (1), § 278,

$$_{10}P_7 = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 604800.$$

2. How many different combinations can be formed with 16 letters, taking 12 at a time?

By § 282, the number of combinations of 16 different things, taken 12 at a time, equals the number of combinations of 16 different things, taken 4 at a time.

Putting $n=16$, $r=4$, in (3), § 280,

$$_{16}C_4 = \frac{16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4} = 1820.$$

3. How many different words, each consisting of 4 consonants and 2 vowels, can be formed from 8 consonants and 4 vowels?

The number of combinations of the 8 consonants, taken 4 at a time, is

$$\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}, \text{ or } 70.$$

The number of combinations of the 4 vowels, taken 2 at a time, is

$$\frac{4 \cdot 3}{1 \cdot 2}, \text{ or } 6.$$

Any one of the 70 sets of consonants may be associated with any one of the 6 sets of vowels; hence, there are in all 70×6, or 420 sets, each containing 4 consonants and 2 vowels.

But each set of 6 letters may have $\underline{|6}$, or 720 different permutations (§ 279).

Therefore, the whole number of different words is

$$420 \times 720, \text{ or } 302400.$$

EXERCISE 70

1. How many different permutations can be formed with 14 letters, taken 6 at a time?

2. In how many different orders can the letters in the word *triangle* be written, taken all together?

3. How many combinations can be formed with 15 things, taken 5 at a time?

4. A certain play has 5 parts, to be taken by a company of 12 persons. In how many different ways can they be assigned?

5. How many combinations can be formed with 17 things, taken 11 at a time?

6. How many different numbers, of 6 different figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number begins with 1, and ends with 9?

7. How many even numbers, of 5 different figures each, can be formed from the digits 4, 5, 6, 7, 8?

8. How many different words, of 8 different letters each, can be formed from the letters in the word *ploughed*, if the third letter is *o*, the fourth *u*, and the seventh *e*?

9. How many different committees, of 8 persons each, can be formed from a corporation of 14 persons? In how many will any particular individual be found?

10. There are 11 points in a plane, no 3 in the same straight line. How many different quadrilaterals can be formed, having 4 of the points for vertices?

11. From a pack of 52 cards, how many different hands of 6 cards each can be dealt?

12. A and B are in a company of 48 men. If the company is divided into equal squads of 6, in how many of them will A and B be in the same squad?

13. How many different words, each having 5 consonants and 1 vowel, can be formed from 13 consonants and 4 vowels?

14. Out of 10 soldiers and 15 sailors, how many different parties can be formed, each consisting of 3 soldiers and 3 sailors?

15. A man has 22 friends, of whom 14 are males. In how many ways can he invite 16 guests from them, so that 10 may be males?

16. From 3 sergeants, 8 corporals, and 16 privates, how many different parties can be formed, each consisting of 1 sergeant, 2 corporals, and 5 privates?

17. Out of 3 capitals, 6 consonants, and 4 vowels, how many different words of 6 letters each can be formed, each beginning with a capital, and having 3 consonants and 2 vowels?

18. How many different words of 8 letters each can be formed from 8 letters, if 4 of the letters cannot be separated? How many if these 4 can only be in one order?

19. How many different numbers, of 7 figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if the first, fourth, and last digits are odd numbers?

284. *To find the number of permutations of n things which are not all different, taken all together.*

Let there be n letters, of which p are a's, q are b's, and r are c's, the rest being all different.

Let N denote the number of permutations of these letters taken all together.

Suppose that, in any particular permutation of the n letters, the p a's were replaced by p new letters, differing from each other and also from the remaining letters.

Then, by simply altering the order of these p letters among themselves, without changing the positions of any of the other letters, we could from the original permutation form $\underline{|p}$ different permutations (§ 279).

If this were done in the case of each of the N original permutations, the whole number of permutations would be $N \times \underline{|p}$.

Again, if in any one of the latter the q b's were replaced by q new letters, differing from each other and from the remaining letters, then by altering the order of these q letters among themselves, we could from the original permutation form $\underline{|q}$ different permutations; and if this were done in the case of each of the $N \times \underline{|p}$ permutations, the whole number of permutations would be $N \times \underline{|p} \times \underline{|q}$.

In like manner, if in each of the latter the r c's were replaced by r new letters, differing from each other and from the remaining letters, and these r letters were permuted among themselves, the whole number of permutations would be

$$N \times \underline{|p} \times \underline{|q} \times \underline{|r}.$$

We now have the original n letters replaced by n *different* letters.

But the number of permutations of n different things taken n at a time is $\underline{|n}$ (§ 279).

Therefore, $N \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n}$; or, $N = \dfrac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$.

Any other case can be treated in a similar manner.

Ex. How many permutations can be formed from the letters in the word *Tennessee*, taken all together?

Here there are 4 e's, 2 n's, 2 s's, and 1 t.
Putting in the above formula $n = 9$, $p = 4$, $q = 2$, $r = 2$, we have

$$\frac{\lfloor 9}{\lfloor 4 \lfloor 2 \lfloor 2} = \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{2 \cdot 2} = 3780.$$

EXERCISE 71

1. In how many different orders can the letters of the word *denomination* be written?

2. There are 4 white billiard balls exactly alike, and 3 red balls, also alike; in how many different orders can they be arranged?

3. In how many different orders can the letters of the word *independence* be written?

4. How many different signals can be made with 7 flags, of which 2 are blue, 3 red, and 2 white, if all are hoisted for each signal?

5. How many different numbers of 8 digits can be formed from the digits 4, 4, 3, 3, 3, 2, 2, 1?

6. In how many different ways can 2 dimes, 3 quarters, 4 halves, and 5 dollars be distributed among 14 persons, so that each may receive a coin?

285. *To find for what value of r the number of combinations of n different things taken r at a time is greatest.*

By § 280, the number of combinations of n different things, taken r at a time, is

$$_nC_r = \frac{n(n-1)\cdots(n-r+2)(n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r}. \qquad (1)$$

Also, the number of combinations of n different things, taken $r-1$ at a time, is

$$\frac{n(n-1)\cdots[n-(r-1)+1]}{1\cdot 2\cdot 3\cdots(r-1)}, \text{ or } \frac{n(n-1)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots r-1}. \qquad (2)$$

The expression (1) is obtained by multiplying the expression (2) by $\frac{n-r+1}{r}$, or $\frac{n+1}{r}-1$.

The latter expression decreases as r increases.

If, then, we find the values of (1) corresponding to the values 1, 2, 3, ⋯, of r, the results will continually increase so long as $\frac{n-r+1}{r}$ is >1.

I. Suppose n even; and let $n=2m$, where m is a positive integer.

Then, $\frac{n-r+1}{r}$ becomes $\frac{2m-r+1}{r}$.

If $r=m$, $\frac{2m-r+1}{r}$ becomes $\frac{m+1}{m}$, and is >1.

If $r=m+1$, $\frac{2m-r+1}{r}$ becomes $\frac{m}{m+1}$, and is <1.

Then, ${}_nC_r$ will have its greatest value when $r=m=\frac{n}{2}$.

II. Suppose n odd; and let $n=2m+1$, where m is a positive integer.

Then, $\frac{n-r+1}{r}$ becomes $\frac{2m-r+2}{r}$.

If $r=m$, $\frac{2m-r+2}{r}$ becomes $\frac{m+2}{m}$, and is >1.

If $r=m+1$, $\frac{2m-r+2}{r}$ becomes $\frac{m+1}{m+1}$, and equals 1.

If $r=m+2$, $\frac{2m-r+2}{r}$ becomes $\frac{m}{m+2}$, and is <1.

Then, ${}_nC_r$ will have its greatest value when r equals m or $m+1$; that is, $\frac{n-1}{2}$ or $\frac{n-1}{2}+1$.

Then, ${}_nC_r$ will have its greatest value when r equals $\frac{n-1}{2}$ or $\frac{n+1}{2}$; the results being the same in these two cases.

XIII. DETERMINANTS

286. The solution of the equations

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

is
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1},\ y = \frac{c_2a_1 - c_1a_2}{a_1b_2 - a_2b_1}.$$

The common denominator may be written in the form

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}. \qquad (1)$$

This is understood as signifying the product of the upper left-hand and lower right-hand numbers, minus the product of the lower left-hand and upper right-hand.

The expression (1) is called a **Determinant of the Second Order.**

The numerators of the above fractions can also be expressed as determinants; thus,

$$b_2c_1 - b_1c_2 = \begin{vmatrix} c_1, & b_1 \\ c_2, & b_2 \end{vmatrix},\ \text{and}\ c_2a_1 - c_1a_2 = \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}.$$

287. The solution of the equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3, \end{cases}$$

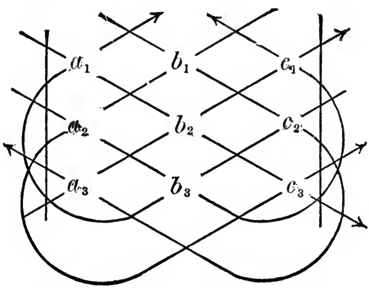
is
$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}; \qquad (1)$$

with results of similar form for y and z.

The denominator of (1) may be written in the form

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}. \qquad (2)$$

This is understood as signifying the sum of the products of the numbers connected by lines parallel to a line joining the upper left-hand corner to the lower right-hand, in the following diagram, minus the sum of the products of the numbers connected by lines parallel to a line joining the lower left-hand corner to the upper right-hand.



The expression (2) is called a **Determinant of the Third Order.**

The numerator of (1) can also be expressed as a determinant, as follows:

$$\begin{vmatrix} d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \\ d_3, & b_3, & c_3 \end{vmatrix};$$

as may be verified by expanding it by the above rule.

EXERCISE 72

Evaluate the following:

1. $\begin{vmatrix} 14 & 15 \\ 9 & 12 \end{vmatrix}$.

2. $\begin{vmatrix} 2x-y & 2x+y \\ 2x+y & 2x-y \end{vmatrix}$.

3. $\begin{vmatrix} 5 & 4 & 3 \\ 2 & 1 & 9 \\ 8 & 7 & 6 \end{vmatrix}$.

4. $\begin{vmatrix} 3 & 4 & 5 \\ 9 & 1 & 2 \\ 6 & 7 & 8 \end{vmatrix}$.

5. $\begin{vmatrix} 15 & 12 & 9 \\ 8 & 4 & 2 \\ 6 & 5 & 4 \end{vmatrix}$.

6. $\begin{vmatrix} 15 & 12 & 9 \\ 8 & 0 & 2 \\ 6 & 5 & 4 \end{vmatrix}$.

7. Show that

$$\begin{vmatrix} 6 & 8 & 10 \\ 7 & 5 & 10 \\ 11 & 6 & 10 \end{vmatrix} = 6\begin{vmatrix} 5 & 10 \\ 6 & 10 \end{vmatrix} - 7\begin{vmatrix} 8 & 10 \\ 6 & 10 \end{vmatrix} + 11\begin{vmatrix} 8 & 10 \\ 5 & 10 \end{vmatrix}.$$

8. $\begin{vmatrix} 8 & 5 & 4 \\ 12 & 9 & 6 \\ 14 & 16 & 7 \end{vmatrix}$.

9. Show that

$$\begin{vmatrix} 7 & -4 & 8 \\ 3 & 6 & -9 \\ 8 & -5 & 13 \end{vmatrix} = -3\begin{vmatrix} -4 & 8 \\ -5 & 13 \end{vmatrix} + 6\begin{vmatrix} 7 & 8 \\ 8 & 13 \end{vmatrix} + 9\begin{vmatrix} 7 & -4 \\ 8 & -5 \end{vmatrix}.$$

It is found in geometry that if the vertices of a triangle are at the points $x = 2$, $y = 3$; $x = 4$, $y = 5$; $x = -1$, $y = 4$, the area of the triangle is found to be

Area $= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \\ -1 & 4 & 1 \end{vmatrix}$, the abscissas (§ 46) forming the first column of the determinant, the ordinates the second column, and the third column being 1's.

Find the areas of the triangles whose vertices are at the following points:

10.* $x=-2$, $y=1$; $x=4$, $y=1$; $x=2$, $y=6$. Make diagram.

11. $x = -4$, $y = 3$; $x = 4$, $y = 3$; $x = 2$, $y = -7$. Make diagram.

12. $x = 2$, $y = 4$; $x = 8$, $y = 4$; $x = -1$, $y = -9$. Make diagram.

13. $x = -5$, $y = 3$; $x = 5$, $y = 3$; $x = 0$, $y = -3$. Make diagram.

14. $x = -2$, $y = 8$; $x = -2$, $y = -2$; $x = 5$, $y = -4$. Make diagram.

* If your area is negative, its absolute value is the area sought. A change in the order of selecting the vertices will change the sign of the area, but not the absolute value of the area.

288. The numbers in the first, second, etc., horizontal lines of a determinant are said to be in the *first, second,* etc., *rows,* respectively; and the numbers in the first, second, etc., vertical columns, in the *first, second,* etc., *columns.*

The numbers constituting the determinant are called its *elements,* and the products in the expanded form its *terms.*

Thus, in the determinant (2), of § 287, the elements are a_1, a_2, a_3, etc., and the terms $a_1b_2c_3$, $-a_1b_3c_2$, etc.

289. If, in any permutation of the numbers 1, 2, 3, ⋯, n, a greater number precedes a less, there is said to be an *inversion.*

Thus, in the case of five numbers, the permutation 4, 3, 1, 5, 2 has six inversions; 4 before 1, 3 before 1, 4 before 2, 3 before 2, 5 before 2, and 4 before 3.

290. General Definition of a Determinant.

Consider the n^2 elements

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & \cdots, & a_{1,n} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & a_{n,2}, & a_{n,3}, & \cdots, & a_{n,n} \end{vmatrix}. \qquad (1)$$

The notation in regard to suffixes, in (1), is that the first suffix denotes the row, and the second the column, in which the element is situated.

Thus, $a_{k,r}$ is the element in the kth row and rth column.

Let all possible products of the elements taken n at a time be formed, subject to the restriction that each product shall contain one and only one element from each row, and one and only one from each column, and write them so that the *first suffixes* shall be in the order 1, 2, 3, ⋯, n.

This is equivalent to writing all the permutations of the numbers 1, 2, 3, ⋯, n in the *second suffixes.*

Make each product + or − according as the number of inversions in the *second* suffixes is *even* or *odd*.

The expression (1) is called a **Determinant of the *n*th Order.**

The number of terms in the expanded form of a determinant of the *n*th order is $\underline{|n}$ (§ 279).

291. The elements lying in the diagonal joining the upper left-hand to the lower right-hand corner, of a determinant, are said to be in the *principal diagonal;* the term whose factors are the elements in the principal diagonal is always positive.

292. It may be shown that the definition of § 290 agrees with that of § 287.

For consider the determinant

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3} \\ a_{2,1}, & a_{2,2}, & a_{2,3} \\ a_{3,1}, & a_{3,2}, & a_{3,3} \end{vmatrix}.$$

The products of the elements taken three at a time, subject to the restriction that each product shall contain one and only one element from each row, and one and only one from each column, the first suffixes being written in the order 1, 2, 3, are

$$a_{1,1}\,a_{2,2}\,a_{3,3},\ a_{1,1}\,a_{2,3}\,a_{3,2},\ a_{1,2}\,a_{2,1}\,a_{3,3},\ a_{1,2}\,a_{2,3}\,a_{3,1},\ a_{1,3}\,a_{2,1}\,a_{3,2},$$
$$\text{and } a_{1,3}\,a_{2,2}\,a_{3,1}.$$

In the first of these there are no inversions in the second suffixes; in the second there is one, 3 before 2; in the third there is one; in the fourth, two; in the fifth, two; in the sixth, three.

Then by the rule of § 290, the first, fourth, and fifth products are positive, and the second, third, and sixth are negative; and the expanded form is

$$a_{1,1}\,a_{2,2}\,a_{3,3} - a_{1,1}\,a_{2,3}\,a_{3,2} - a_{1,2}\,a_{2,1}\,a_{3,3} + a_{1,2}\,a_{2,3}\,a_{3,1}$$
$$+ a_{1,3}\,a_{2,1}\,a_{3,2} - a_{1,3}\,a_{2,2}\,a_{3,1},$$

which agrees with § 287.

293. The expanded form of the determinant (1), § 290, may also be obtained by writing the *second suffixes* in the order 1, 2, 3, ⋯, n, and making each product + or − according as the number of inversions in the *first suffixes* is even or odd.

For let the absolute value of any term, obtained by the rule of § 290, be

$$a_{1,p}\,a_{2,q}\,\cdots\,a_{n,r}\,; \qquad (1)$$

where $p, q, \cdots, r$ is a permutation of 1, 2, ⋯, n.

This is obtained from the first term

$$a_{1,1}\,a_{2,2}\,\cdots\,a_{n,n} \qquad (2)$$

by changing second suffixes, 1 to p, 2 to q, ⋯, n to r.

Since $p, q, \cdots, r$ is a permutation of 1, 2, ⋯, n, (2) may be written

$$a_{p,p}\,a_{q,q}\,\cdots\,a_{r,r}\,;$$

and (1) may be obtained from this by changing *first* suffixes, p to 1, q to 2, ⋯, r to n.

In these two ways, we have the same number of interchanges of two suffixes, and hence the term (1) will have the same sign.

PROPERTIES OF DETERMINANTS

294. A determinant is not altered in value if its rows are changed to columns, and its columns to rows.

Consider the determinants

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & \cdots, & a_{1,n} \\ a_{2,1}, & a_{2,2}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & a_{n,2}, & \cdots, & a_{n,n} \end{vmatrix} \text{ and } \begin{vmatrix} a_{1,1}, & a_{2,1}, & \cdots, & a_{n,1} \\ a_{1,2}, & a_{2,2}, & \cdots, & a_{n,2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1,n}, & a_{2,n}, & \cdots, & a_{n,n} \end{vmatrix}.$$

Since the second suffixes of the first determinant are the same as the first suffixes of the second, if the first determinant be expanded by the rule of § 290, and the second by the rule of § 293, the results will be the same.

Therefore the determinants are equal.

295. A determinant is changed in sign if any two consecutive rows, or any two consecutive columns, are interchanged.

Consider the determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & k \end{vmatrix}.$$

Evaluating each determinant as in Exercise 72, we have

$$aek + dhc + bfg - gec - bdk - fha$$

and
$$bdk + egc + afh - hdc - aek - fgb,$$

each term of the second determinant being the negative of the corresponding term in the first. The second determinant has therefore the same absolute value as the first, but of opposite sign. In a manner similar to that used in § 294, it may be shown that this property holds for determinants of the nth order.

It follows, from §§ 294 and 295, that if two consecutive rows are interchanged, the sign of the determinant is changed.

296. A determinant is changed in sign if any two rows, or any two columns, are interchanged.

Consider the m letters $a, b, c, \cdots, e, f, g$.

By interchanging a with b, then a with c, and so on in succession with each of the $m - 1$ letters to the right of a, a may be brought to the right of g.

Then, by interchanging g with f, then g with e, and so on in succession with each of the $m - 2$ letters to the left of g, g may be brought to the left of b.

That is, a and g may be interchanged by $(m - 1) + (m - 2)$, or $2m - 3$, interchanges of consecutive letters; that is, by an *odd number* of interchanges of consecutive letters.

It follows from the above that any two rows, or any two columns, of a determinant may be interchanged by an *odd number* of interchanges of consecutive rows or columns.

But every interchange of two consecutive rows or columns changes the sign of the determinant (§ 295).

Therefore the sign of the determinant is changed if any two rows, or any two columns, are interchanged.

297. Cyclical interchange of rows or columns.

By $n-1$ successive interchanges of two consecutive rows, the first row of a determinant of the nth order may be made the last.

Thus, by § 295, the determinant

$$\begin{vmatrix} a_1, & b_1, & \cdots, & m_1 \\ a_2, & b_2, & \cdots, & m_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n, & b_n, & \cdots, & m_n \end{vmatrix} \text{ is equal to } (-1)^{n-1} \begin{vmatrix} a_2, & b_2, & \cdots, & m_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n, & b_n, & \cdots & m_n \\ a_1, & b_1, & \cdots, & m_1 \end{vmatrix}.$$

The above is called a *cyclical* interchange of rows.

In like manner, by $n-1$ successive interchanges of two consecutive columns, the first column of a determinant of the nth order may be made the last.

298. If two rows, or two columns, of a determinant are identical, the value of the determinant is zero.

Let D be the value of a determinant having two rows, or two columns, identical.

If these rows, or columns, are interchanged, the value of the resulting determinant is $-D$ (§ 296).

But since the rows, or columns, which are interchanged are identical, the two determinants are of equal value.

Hence, $D=-D$; and this equation cannot be satisfied by any value of D except 0.

Ex. Evaluate the following determinant:

$$\begin{vmatrix} a & a & d \\ b & b & e \\ c & c & k \end{vmatrix} = abk + bcd + cea - cbd - abk - ace.$$
$$= 0.$$

299. If each element in one column, or in one row, is the sum of m terms, the determinant can be expressed as the sum of m determinants.

Consider the determinant

$$\begin{vmatrix} a_{1,1}, & \cdots, & a_{1,r}, & \cdots, & a_{1,n} \\ a_{2,1}, & \cdots, & a_{2,r}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & a_{n,r}, & \cdots, & a_{n,n} \end{vmatrix}. \qquad (1)$$

Let each element in the rth column be the sum of m terms, as follows:

$$\begin{aligned} a_{1,r} &= b_1 + c_1 + \cdots + f_1, \\ a_{2,r} &= b_2 + c_2 + \cdots + f_2, \\ &\cdots\cdots\cdots \\ a_{n,r} &= b_n + c_n + \cdots + f_n. \end{aligned}$$

Let $a_{1,p} \cdots a_{q,r} \cdots a_{n,s}$ be the absolute value of one of the terms of (1); then

$$\begin{aligned} a_{1,p} \cdots a_{q,r} \cdots a_{n,s} &= a_{1,p} \cdots (b_q + c_q = \cdots + f_q) \cdots a_{n,s} \\ &= (a_{1,p} \cdots b \ \cdots a_{n,s}) + \cdots + (a_{1,p} \cdots f_q \cdots a_{n,s}). \end{aligned}$$

It is evident from this that the determinant (1) can be expressed as the sum of the determinants

$$\begin{vmatrix} a_{1,1}, & \cdots, & b_1, & \cdots, & a_{1,n} \\ a_{2,1}, & \cdots, & b_2, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & b_n, & \cdots, & a_{n,n} \end{vmatrix} + \cdots + \begin{vmatrix} a_{1,1}, & \cdots, & f_1, & \cdots, & a_{1,n} \\ a_{2,1}, & \cdots, & f_2, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & f_n, & \cdots, & a_{n,n} \end{vmatrix}.$$

300. If all the elements in one column, or in one row, are multiplied by the same number, the determinant is multiplied by this number.

Consider the determinant

$$\begin{vmatrix} a_{1,1}, & \cdots, & a_{1,r}, & \cdots, & a_{1,n} \\ a_{2,1}, & \cdots, & a_{2,r}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & a_{n,r}, & \cdots & a_{n,n} \end{vmatrix}. \qquad (1)$$

Multiplying each element in the rth column by m, we have

$$\begin{vmatrix} a_{1,1}, & \cdots, & ma_{1,r}, & \cdots, & a_{1,n} \\ a_{2,1}, & \cdots, & ma_{2,r}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & ma_{n,r}, & \cdots & a_{n,n} \end{vmatrix}. \qquad (2)$$

Let $a_{1,p} \cdots a_{q,r} \cdots a_{n,s}$ be the absolute value of one of the terms of (1). Replacing $a_{q,r}$ by $ma_{q,r}$, the absolute value of the corresponding term of (2) is $ma_{1,p} \cdots a_{q,r} \cdots a_{n,s}$.

It is evident from this that the determinant (2) equals m times the determinant (1).

Ex. Consider the values of

$$m\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} ma & d & g \\ mb & e & h \\ mc & f & k \end{vmatrix}.$$

Evaluating, $m(aek + bfg + chd - ceg - dbk - afh)$ and $maek + mbfg + mchd - mceg - dmbk - mafh$, which are identical.

301. If all the elements in any column, or row, be multiplied by the same number, and either added to, or subtracted from, the corresponding elements in another column, or row, the value of the determinant is not changed.

Let the elements in the rth column of the following determinant be multiplied by m, and added to the corresponding elements in the qth column.

$$\begin{vmatrix} a_1, & \cdots, & a_q, & \cdots, & a_r, & \cdots, & a_n \\ b_1, & \cdots, & b_q, & \cdots, & b_r, & \cdots, & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1, & \cdots, & k_q, & \cdots, & k_r, & \cdots, & k_n \end{vmatrix}. \qquad (1)$$

We then obtain the determinant

$$\begin{vmatrix} a_1, & \cdots, & a_q + ma_r, & \cdots, & a_r, & \cdots, & a_n \\ b_1, & \cdots, & b_q + mb_r, & \cdots, & b_r, & \cdots, & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1, & \cdots, & k_q + mk_r, & \cdots, & k_r, & \cdots, & k_n \end{vmatrix}. \qquad (2)$$

which by §§ 299 and 300, is equal to

$$\begin{vmatrix} a_1, & \cdots, & a_q, & \cdots, & a_r, & \cdots, & a_n \\ b_1, & \cdots, & b_q, & \cdots, & b_r, & \cdots, & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1, & \cdots, & k_q, & \cdots, & k_r, & \cdots, & k_n \end{vmatrix} + m \begin{vmatrix} a_1, & \cdots, & a_r, & \cdots, & a_r, & \cdots, & a_n \\ b_1, & \cdots, & b_r, & \cdots, & b_r, & \cdots, & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1, & \cdots, & k_r, & \cdots, & k_r, & \cdots, & k_n \end{vmatrix}.$$

But the coefficient of m is zero (§ 298).

Whence, the determinant (2) is equal to (1).

302. Minors.

If the elements in any m rows and any m columns of a determinant of the nth order be erased, the remaining elements form a determinant of the $(n-m)$th order.

This determinant is called an ***m*th Minor** of the given determinant.

$$\text{Thus, } \begin{vmatrix} a_1, & d_1, & e_1 \\ a_3, & d_3, & e_3 \\ a_5, & d_5, & e_5 \end{vmatrix} \text{ is a second minor of } \begin{vmatrix} a_1, & b_1, & c_1, & d_1, & e_1 \\ a_2, & b_2, & c_2, & d_2, & e_2 \\ a_3, & b_3, & c_3, & d_3, & e_3 \\ a_4, & b_4, & c_4, & d_4, & e_4 \\ a_5, & b_5, & c_5, & d_5, & e_5 \end{vmatrix},$$

obtained by erasing the second and fourth rows, and the second and third columns.

303. *To find the coefficient of $a_{1,1}$ in the determinant*

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & \cdots, & a_{1,n} \\ a_{2,1}, & a_{2,2}, & \cdots, & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & a_{n,2}, & \cdots, & a_{n,n} \end{vmatrix}. \qquad (1)$$

By § 290, the absolute values of the terms which involve $a_{1,1}$ are obtained by forming all possible products of the elements taken n at a time, subject to the restrictions that the first elements shall be $a_{1,1}$ and that each product shall contain one and only one element from each row except the first, and one and only one from each column except the first.

It is evident from this that the *coefficient* of $a_{1,1}$ in (1) may be obtained by forming all possible products of the following elements taken $n-1$ at a time,

$$\begin{matrix} a_{2,2}, & a_{2,3}, & \cdots, & a_{2,n} \\ a_{3,2}, & a_{3,3}, & \cdots, & a_{3,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,2}, & a_{n,3}, & \cdots, & a_{n,n} \end{matrix}$$

subject to the restriction that each product shall contain one and only one element from each row, and one and only one from each column, writing the first suffixes in the order $2, 3, \cdots, n$, and making each product $+$ or $-$ according as the number of inversions in the second suffixes is even or odd.

Then by § 290, the coefficient of $a_{1,1}$ is

$$\begin{vmatrix} a_{2,2}, & a_{2,3}, & \cdots, & a_{2,n} \\ a_{3,2}, & a_{3,3}, & \cdots, & a_{3,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,2}, & a_{n,3}, & \cdots, & a_{n,n} \end{vmatrix};$$

that is, the minor obtained by erasing the first row and the first column of the given determinant.

304. By aid of § 303, a determinant of any order may be expressed as a determinant of any higher order.

$$\text{Thus,}\quad \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & a_1, & b_1, & c_1 \\ 0, & a_2, & b_2, & c_2 \\ 0, & a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & a_1, & b_1, & c_1 \\ 0, & 0, & a_2, & b_2, & c_2 \\ 0, & 0, & a_3, & b_3, & c_3 \end{vmatrix}, \text{ etc.}$$

305. Coefficient of any Element of a Determinant.

To find the coefficient of b_3, in the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}. \qquad (1)$$

By two interchanges of consecutive rows, the last row may be made the first; thus, by § 295, the determinant equals

$$(-1)^2 \begin{vmatrix} a_3, & b_3, & c_3 \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix}. \qquad (2)$$

By interchanging the first two columns, the determinant (2) equals

$$(-1)^3 \begin{vmatrix} b_3, & a_3, & c_3 \\ b_1, & a_1, & c_1 \\ b_2, & a_2, & c_2 \end{vmatrix}. \qquad (3)$$

By § 303, the coefficient of b_3, in (3), is

$$(-1)^{3+2} \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}.$$

That is, the coefficient of the element in the third row and second column equals $(-1)^{3+2}$ multiplied by that minor of (1) which is obtained by erasing the third row and second column.

We will now consider the general case; to find the coefficient of $a_{k,r}$ in the determinant

$$\begin{vmatrix} a_{1,1}, & \cdots, & a_{1,r}, & \cdots, & a_{1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k,1}, & \cdots, & a_{k,r}, & \cdots, & a_{k,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & a_{n,r}, & \cdots, & a_{n,n} \end{vmatrix}. \qquad (4)$$

By $k-1$ interchanges of consecutive rows, and $r-1$ interchanges of consecutive columns, the element $a_{k,r}$ may be brought to the upper left-hand corner.

Thus, by § 295, the determinant equals

$$(-1)^{k-1}(-1)^{r-1}\begin{vmatrix} a_{k,r}, & a_{k,1}, & \cdots, & a_{k,n} \\ a_{1,r}, & a_{1,1}, & \cdots, & a_{1,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,r}, & a_{n,1}, & \cdots, & a_{n,n} \end{vmatrix}.$$

Then, by § 303, the coefficient of $a_{k,r}$ is

$$(-1)^{k+r-2}\begin{vmatrix} a_{1,1}, & \cdots, & a_{1,n} \\ \cdot & \cdot & \cdot \\ a_{n,1}, & \cdots, & a_{n,n} \end{vmatrix}.$$

But $\qquad (-1)^{k+r-2}=(-1)^{k+r}\div(-1)^2=(-1)^{k+r}.$

Hence, the coefficient of the element in the kth row and rth column equals $(-1)^{k+r}$, multiplied by the minor of (4) which is obtained by erasing the kth row and rth column.

306. By aid of § 305, a determinant of any order may be expressed in terms of determinants of any lower order.

Thus, since every term of a determinant contains one and only one element from the first row, we have,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \\ a_4, & c_4, & d_4 \end{vmatrix}$$

$$+ c_1\begin{vmatrix} a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \\ a_4, & b_4, & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \\ a_4, & b_4, & c_4 \end{vmatrix};$$

and each of the latter determinants may in turn be expressed in terms of determinants of the second order.

307. Evaluation of Determinants.

The method of § 306 may be used to express a determinant of any order higher than the third in terms of determinants of the third order, which may be evaluated by the rule of § 290.

The theorem of § 301 may often be employed to shorten the process.

1. Evaluate $\begin{vmatrix} 5, & 7, & 8, & 6 \\ 11, & 16, & 13, & 11 \\ 14, & 24, & 20, & 23 \\ 7, & 13, & 12, & 2 \end{vmatrix}$.

Subtracting the first row from the last, twice the first row from the second, and three times the first row from the third (§ 301), the determinant becomes

$$\begin{vmatrix} 5, & 7, & 8, & 6 \\ 1, & 2, & -3, & -1 \\ -1, & 3, & -4, & 5 \\ 2, & 6, & 4, & -4 \end{vmatrix} = 2 \begin{vmatrix} 5, & 7, & 8, & 6 \\ 1, & 2, & -3, & -1 \\ -1, & 3, & -4, & 5 \\ 1, & 3, & 2, & -2 \end{vmatrix}, \text{ by § 300}$$

Subtracting five times the second row from the first, adding the second row to the third, and subtracting the second row from the last, we have

$$2 \begin{vmatrix} 0, & -3, & 23, & 11 \\ 1, & 2, & -3, & -1 \\ 0, & 5, & -7, & 4 \\ 0, & 1, & 5, & -1 \end{vmatrix} = -2 \begin{vmatrix} -3, & 23, & 11 \\ 5, & -7, & 4 \\ 1, & 5, & -1 \end{vmatrix}, \text{ by § 305.}$$

The object of the above process is to put the given determinant in such a form that all but one of the elements in one column shall be zero; the determinant can then be expressed as a determinant of the third order by § 305.

The last determinant may be evaluated by § 287; but it is better to subtract five times the first column from the second, and then add the first column to the last; thus,

$$-2 \begin{vmatrix} -3, & 38, & 8 \\ 5, & -32, & 9 \\ 1, & 0, & 0 \end{vmatrix} = -2 \begin{vmatrix} 38, & 8 \\ -32, & 9 \end{vmatrix} = -2\,(342 + 256) = -1196.$$

The artifice used in the following example is often of use in evaluation of determinants:

2. Evaluate $\begin{vmatrix} x, & x^2, & x^3 \\ y, & y^2, & y^3 \\ z, & z^2, & z^3 \end{vmatrix}$.

If x be put equal to y, the determinant has two rows identical, and equals zero (§ 298).

Then, $x - y$ must be a factor (§ 105); and in like manner, $y - z$ and $z - x$ are factors.

Let the given determinant $= X(x-y)(y-z)(z-x)$.

To determine X, we observe that x, y, and z are factors of the determinant; then, X must equal xyz, as it is evident by noticing that the first term in the expanded form is $+ xy^2z^3$, and the value of the determinant is $xyz(x-y)(y-z)(z-x)$.

EXERCISE 73

Evaluate the following:

1. $\begin{vmatrix} 8, & 24, & 16 \\ 3, & 9, & 15 \\ 21, & 30, & 26 \end{vmatrix}$.

2. $\begin{vmatrix} 12, & 5, & 6 \\ 14, & 41, & 7 \\ 16, & 39, & 8 \end{vmatrix}$.

3. $\begin{vmatrix} 13, & 14, & 15 \\ 14, & 15, & 13 \\ 15, & 13, & 14 \end{vmatrix}$.

4. $\begin{vmatrix} a+b, & a, & b \\ b+c, & b, & c \\ c+a, & c, & a \end{vmatrix}$.

5. $\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x, & y, & 1 \end{vmatrix}$.

6. $\begin{vmatrix} x, & y, & 1 \\ 3, & 5, & 1 \\ 6, & -12, & 1 \end{vmatrix} = 0.$

Plot this line (§ 51). Are the points $x = 3$, $y = 5$, and $x = 6$, $y = -12$, on this line?

7. $\begin{vmatrix} x, & y, & 1 \\ -4, & -7, & 1 \\ 8, & 6, & 1 \end{vmatrix} = 0.$

Are the points $x = -4$, $y = -7$, and $x = 8$, $y = 6$, on this line?

8. $\begin{vmatrix} 15, & 12, & 11, & 6 \\ 14, & -4, & -8, & 25 \\ 16, & 10, & -2, & 5 \\ 10, & 12, & 3, & 6 \end{vmatrix}$.

9. $\begin{vmatrix} 4, & 0, & 0, & 3 \\ 7, & 9, & 0, & 0 \\ 8, & 0, & 0, & 10 \\ 0, & 0, & -6, & -8 \end{vmatrix}$.

10. $\begin{vmatrix} 0, & 3, & 5, & 9 \\ 5, & 0, & 7, & 8 \\ 4, & 9, & 0, & 16 \\ 8, & 16, & 4, & 0 \end{vmatrix}$.

11. $\begin{vmatrix} 4, & 0, & 12, & 10 \\ 0, & 3, & 0, & 0 \\ 10, & 0, & 6, & 4 \\ 24, & 0, & 3, & 12 \end{vmatrix}$.

12. $\begin{vmatrix} 8, & 0, & 0, & 0 \\ 0, & 6, & 15, & 18 \\ 0, & 12, & 98, & 104 \\ 0, & 2, & 5, & 6 \end{vmatrix}$.

13. Show that $\begin{vmatrix} 5, & -4, & 0, & 15 \\ 12, & 0, & 5, & 4 \\ 17, & -9, & -9, & 3 \\ 6, & 8, & 8, & -2 \end{vmatrix} = \begin{vmatrix} -4, & 5, & 15, & 0 \\ 0, & 12, & 4, & 5 \\ -9, & 17, & 3, & -9 \\ 8, & 6, & -2, & 8 \end{vmatrix}$.

14. Show that $\begin{vmatrix} 14, & 3, & -13, & 1 \\ 8, & -5, & 21, & 2 \\ 21, & 24, & -17, & 1 \\ -6, & -9, & -11, & 2 \end{vmatrix} = \begin{vmatrix} -13, & 21, & -17, & -11 \\ 14, & 8, & 21, & -6 \\ 3, & -5, & 24, & -9 \\ 1, & 2, & 1, & 2 \end{vmatrix}$.

15. Show that $\begin{vmatrix} 5, & 6, & 12, & 16 \\ 7, & 9, & -16, & -15 \\ 18, & 27, & 36; & 9 \\ 2, & 9, & -20, & -11 \end{vmatrix} = 108 \begin{vmatrix} 5, & 2, & 3, & 16 \\ 7, & 3, & -4, & -15 \\ 2, & 1, & 1, & 1 \\ 2, & 3, & -5, & -11 \end{vmatrix}$.

16. Evaluate $\begin{vmatrix} 16, & 8, & 7, & -4 \\ 8, & 6, & 3, & 2 \\ 7, & 5, & -11, & 3 \\ -4, & 7, & 8, & 0 \end{vmatrix}$.

17. Evaluate $\begin{vmatrix} m, & y, & n, & x \\ x, & y, & n, & m \\ x, & n, & y, & m \\ m, & n, & y, & x \end{vmatrix}$.

It is shown in geometry that if a straight line passes through the points $x = x_1$, $y = y_1$, $x = x_2$, $y = y_2$, where x_1, x_2, y_1, y_2 are definite values of x and y, the equation of the line is found from the determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Find the equations of the lines passing through the following points:

18. $x = 1,\ y = 3;\ x = -2,\ y = 4.$

19. $x = 0,\ y = 8;\ x = 8,\ y = 0.$

20. $x = -1,\ y = 2;\ x = 3,\ y = -1.$

308. Let A_r, B_r, $\cdots$, K_r, denote the coefficients of the elements a_r, b_r, $\cdots$, k_r, respectively, in the determinant.

$$\begin{vmatrix} a_1, & b_1, & \cdots, & k_1 \\ a_2, & b_2, & \cdots, & k_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n, & b_n, & \cdots, & k_n \end{vmatrix}. \qquad (1)$$

Then, since every term of the determinant contains one and only one element from the first column, the value of the determinant is

$$A_1a_1 + A_2a_2 + \cdots + A_na_n.$$

In like manner, the value of the determinant also equals

$$B_1b_1 + B_2b_2 + \cdots + B_nb_n, \cdots, K_1k_1 + K_2k_2 + \cdots + K_nk_n.$$

309. If m_1, m_2, $\cdots$, m_n are the elements in *any column* of the determinant (1), of § 308, except the first,

$$A_1m_1 + A_2m_2 + \cdots + A_nm_n$$

is the value of a determinant, which differs from (1) only in having m_1, m_2, $\cdots$, m_n instead of a_1, a_2, $\cdots$, a_n as the elements in the first column.

Then $$A_1m_1 + A_2m_2 + \cdots + A_nm_n = 0;$$

for it is the value of a determinant which has two columns identical.

In like manner, if m_1, m_2, $\cdots$, m_n are the elements in any column of the determinant (1), except the second,

$$B_1m_1 + B_2m_2 + \cdots + B_nm_n = 0;$$

and so on.

SOLUTION OF EQUATIONS

310. Let it be required to solve the following system of n linear, simultaneous equations, involving n unknown numbers:

$$\begin{cases} a_1x_1 + \cdots + q_1x_r + \cdots + k_1x_n = p_1. & (1) \\ a_2x_1 + \cdots + q_2x_r + \cdots + k_2x_n = p_2. & (2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ a_nx_1 + \cdots + q_nx_r + \cdots + k_nx_n = p_n. & (3) \end{cases}$$

Let $Q_1, Q_2, \cdots, Q_n$ denote the coefficients of the elements $q_1, q_2, \cdots, q_n$, respectively, in the determinant

$$D = \begin{vmatrix} a_1, & \cdots, & q_1, & \cdots, & k_1 \\ a_2, & \cdots, & q_2, & \cdots, & k_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n, & \cdots, & q_n, & \cdots, & k_n \end{vmatrix}.$$

Multiplying equations (1), (2), $\cdots$, (3) by $Q_1, Q_2, \cdots, Q_n$, respectively, and adding, we have

$$\begin{aligned} &x_1(Q_1a_1 + Q_2a_2 + \cdots + Q_na_n) + \cdots \\ &+ x_r(Q_1q_1 + Q_2q_2 + \cdots + Q_nq_n) + \cdots \\ &+ x_n(Q_1k_1 + Q_2k_2 + \cdots + Q_nk_n) = Q_1p_1 + Q_2p_2 + \cdots + Q_np_n. \end{aligned}$$

By § 309 the coefficient of each unknown number, except x_r, is zero.

By § 308, the coefficient of x_r is D; also, the second member is the value of a determinant which differs from D only in having $p_1, p_2, \cdots, p_n$ instead of $q_1, q_2, \cdots, q_n$ as the elements in the rth column; denoting the latter by D_r, we have

$$x_rD = D_r, \text{ and } x_r = \frac{D_r}{D}.$$

311. *Ex.* Find the value of y from the equations

$$\begin{cases} 3x - 5y + 7z = 28. \\ 2x + 6y - 9z = -23. \\ 4x - 2y - 5z = 9. \end{cases}$$

The denominator of the value of y is the determinant

$$\begin{vmatrix} 3, & -5, & 7 \\ 2, & 6, & -9 \\ 4, & -2, & -5 \end{vmatrix}.$$

The numerator is obtained by putting for the second column the second members of the given equations.

Therefore,
$$y=\frac{\begin{vmatrix}3, & 28, & 7\\ 2, & -23, & -9\\ 4, & 9, & -5\end{vmatrix}}{\begin{vmatrix}3, & -5, & 7\\ 2, & 6, & -9\\ 4, & -2, & -5\end{vmatrix}}=\frac{630}{-210}=-3.$$

EXERCISE 74

Solve by determinants, checking each result:

1. $\begin{cases}2x+7y=-39.\\5x+2y=11.\end{cases}$

2. $\begin{cases}15x+11y=82.\\9x-14y=8.\end{cases}$

3. $\begin{cases}2x-y+3z=9.\\x+2y-z=2.\\3x+5y-4z=1.\end{cases}$

4. $\begin{cases}5x-7y+z=-1.\\2x+8y-7z=-26.\\x+2y+4z=4.\end{cases}$

5. $\begin{cases}6x-4y-7z=17.\\9x-7y-16z=29.\\10x-5y-3z=23.\end{cases}$

6. $\begin{cases}3x+5y=1.\\9x+5z=-7.\\9y+3z=2.\end{cases}$

7. $\begin{cases}4x-11y-5z=9.\\8x+4y-z=11.\\16x+7y+6z=64.\end{cases}$

8. $\begin{cases}2x-y+z=-9.\\x-2y+z=0.\\x-y+2z=-11.\end{cases}$

9. $\begin{cases}x+2y-3z=5.\\3x-22y+6z=4.\\7x-6y-3z=15.\end{cases}$

10. $\begin{cases}\frac{x}{a}+\frac{y}{c}=2b.\\\frac{x}{b}+\frac{z}{c}=2a.\\\frac{y}{b}+\frac{z}{a}=2c.\end{cases}$

XIV. THEORY OF EQUATIONS

312. Every equation of the nth degree, with one unknown number, can be written in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0; \qquad (1)$$

where the coefficients may be positive or negative, integral or fractional, rational or irrational, real or imaginary, or zero.

If no coefficient equals zero, the equation is said to be *Complete;* otherwise it is said to be *Incomplete.*

We shall call (1) the *General Form* of the equation of the nth degree.

313. We assume that every equation of the above form has at least one root, real or imaginary.

314. Divisibility of Equations.

It follows, from § 105, that if the equation

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

has a as a root, then the first member is divisible by $x - a$.

For, if a is a root, the first member becomes 0 when x is put equal to a.

315. (Converse of § 314.) *If the first member of the equation*

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

is divisible by $x - a$, then a is a root of the equation.

For since the first member of the given equation is divisible by $x - a$, the equation may be put in the form

$$(x - a)Q = 0;$$

and it follows from § 110 that a is a root of this equation.

It follows from the above that if the first member of

$$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

is divisible by $ax + b$, then $-\frac{b}{a}$ is a root of the equation.

316. Number of Roots.

An equation of the nth degree has n roots, and not more than n.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

By § 313, this equation has at least one root.

Let a be this root; then, by § 314, the first member is divisible by $x - a$, and the equation may be put in the form

$$(x-a)(x^{n-1} + q_2x^{n-2} + \cdots + q^{n-1}x + q_n) = 0.$$

By § 110, the latter equation may be solved by placing

$$x - a = 0,$$

and $$x^{n-1} + q_2x^{n-2} + \cdots + q_{n-1}x + q_n = 0. \qquad (2)$$

Equation (2) must also have at least one root.

Let b be this root; then (2) may be written

$$(x-b)(x^{n-2} + r_3x^{n-3} + \cdots + r_{n-1}x + r_n) = 0,$$

and the equation may be solved by placing

$$x - b = 0,$$

and $$x^{n-2} + r_3x^{n-3} + \cdots + r_{n-1}x + r_n = 0.$$

After $n - 1$ binomial factors have been divided out, we shall arrive finally at an equation of the first degree,

$$x - k = 0; \text{ whence, } x = k.$$

Therefore, the given equation has the n roots $a, b, \cdots, k$.

The roots are not necessarily *unequal*.

Ex. $$x^3 - 3x^2 + 3x - 1 = 0.$$

Whence, $$(x-1)(x-1)(x-1) = 0$$

and $$x = 1, \text{ or } 1, \text{ or } 1.$$

317. Depression of Equations.

It follows from § 316 that, if m roots of an equation of the nth degree are known, the equation may be depressed to another equation of the $(n-m)$th degree, which shall contain the other $n-m$ roots.

Thus, if all but two of the roots of an equation are known, these two may be obtained from the depressed equation by the rules for quadratics.

Ex. Two roots of the equation $9x^4 - 37x^2 - 8x + 20 = 0$ are 2 and $-\frac{5}{3}$; find the others.

By § 314, the first member of the given equation is divisible by $(x-2)(3x+5)$, or $3x^2 - x - 10$.

Dividing $9x^4 - 37x^2 - 8x + 20$ by $3x^2 - x - 10$, the quotient is $3x^2 + x - 2$.

Then the depressed equation is

$$3x^2 + x - 2 = 0.$$

Solving by the rules for quadratics, $x = \frac{2}{3}$ or -1.

EXERCISE 75

Find whether:

1. One root of $x^3 - x^2 - 32x + 60 = 0$ is 5; if so, find the others.

2. One root of $2x^3 - 6x^2 - 2x + 6 = 0$ is 3; if so, find the others.

3. Two roots of $4x^4 - 12x^3 - 13x^2 + 45x - 18 = 0$ are -2 and 3 respectively; if so, find the others.

4. One root of $3x^3 - 5x^2 + 2x - 4 = 0$ is 2; if so, find the others.

5. Two roots of $14x^4 + 65x^3 - 222x^2 + 65x + 14 = 0$ are 2 and -7; if so, find the others.

6. Two roots of $x^4 + 4x^3 - 6x^2 + 24x - 72 = 0$ are -6 and 2; if so, find the others.

7. Two roots of $x^4 - 4x^3 + 3x^2 + 4x - 4 = 0$ are 2 and -1; if so, find the others.

8. Three roots of $2x^5 + 29x^4 + 156x^3 + 379x^2 + 394x + 120 = 0$ are -2, -3, $-\frac{1}{2}$; if so, find the others.

9. Two roots of $x^4 + 12x^3 + 34x^2 - 12x - 35 = 0$ are 1 and -7; if so, find the others.

10. Two roots of $5x^4 - 18x^3 + 72x - 120 = 0$ are 5 and -4; if so, find the others.

318. Formation of Equations.

It follows from § 316, that if the roots of

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

are $a, b, \cdots, k$, the equation may be written in the form

$$(x-a)(x-b)\cdots(x-k) = 0.$$

Hence, to form an equation which shall have any required roots,

Subtract each root from x, and place the product of the resulting expressions equal to zero.

Ex. Form an equation having the roots 1, $\frac{1}{2}$, and $-\frac{5}{3}$.

By the rule $(x-1)(x-\frac{1}{2})(x+\frac{5}{3}) = 0.$

Multiplying the terms of the second factor by 2, and of the third by 3,

$$(x-1)(2x-1)(3x+5) = 0.$$

Expanding, $6x^3 + x^2 - 12x + 5 = 0.$

EXERCISE 76

Form equations having the roots:

1. $1, -4, 6.$
2. $3, -\frac{1}{2}, -2\frac{1}{2}.$
3. $2, 3, 5, 0.$
4. $-1, -2, 7, -8.$
5. $2, 9, -\frac{1}{3}, \frac{5}{2}.$
6. $4, 4, -\frac{3}{4}, -\frac{3}{4}.$
7. $-m, \dfrac{m \pm \sqrt{n}}{4}.$
8. $3 \pm \sqrt{2}, -3 \pm \sqrt{2}.$
9. $a, -\dfrac{1}{a}, -b, -\dfrac{1}{b}.$
10. $\dfrac{4 \pm 2\sqrt{3}}{3}, \dfrac{-2 \pm \sqrt{3}}{3}.$

319. Composition of Coefficients.

By § 318, the equation of the nth degree whose roots are a, b, c, d, $\cdots$, k, l, m, is

$$(x-a)(x-b)(x-c)(x-d)\cdots(x-m)=0. \qquad (1)$$

By actual multiplication, we obtain

$$(x-a)(x-b)=x^2-(a+b)x+ab;$$

$$(x-a)(x-b)(x-c)$$
$$=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc;$$

$$(x-a)(x-b)(x-c)(x-d)$$
$$=x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$-(abc+abd+acd+bcd)x+abcd=0;\text{ etc.}$$

When all the factors of the first member of (1) have been multiplied together, the result will be in the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\cdots+p_n;$$

where
$$p_1=-(a+b+c+\cdots+k+l+m);$$
$$p_2=ab+ac+bc+\cdots+lm;$$
$$p_3=-(abc+abd+acd+\cdots+klm);$$
. . . .
$$p_n=\pm\, abcd\cdots klm, \text{ according as } n \text{ is even or odd.}$$

Hence, in an equation of the nth degree in the *general form*,

The coefficient of the second term is equal to minus the sum of all the roots.

The coefficient of the third term is equal to the sum of the products of the roots, taken two at a time,

The coefficient of the fourth term is equal to minus the sum of the products of the roots, taken three at a time; etc.

The last term is equal to plus or minus the product of all the roots, according as n is even or odd.

320. It follows from § 319 that, if an equation of the nth degree is in the general form,

If the second term is wanting, the sum of the roots is 0.

If the last term is wanting, at least one root is 0.

If the last term is an integer, it is divisible by every integral root.

EXERCISE 77

In each of the following, find the sum of the roots, and the product of the roots:

1. $x^3 - 8x^2 + 19x - 12 = 0.$
2. $x^3 - 31x - 30 = 0.$
3. $4x^4 - 12x^3 + 3x^2 + 13x - 6 = 0.$

321. If all but one of the roots of an equation of the nth degree in the general form are known, the remaining root may be found by changing the sign of the coefficient of the second term of the given equation, and subtracting the sum of the known roots from the result; or, by dividing the last term of the given equation if n is even, or its negative if n is odd, by the product of the known roots.

If all but two are known, the coefficient of the second term of the depressed equation may be found by adding the sum of the known roots to the coefficient of the second term of the given equation; and the last term of the depressed equation may be found by dividing the last term of the given equation by plus or minus the product of the known roots according as n is even or odd.

Ex. Two roots of the equation $9x^4 - 37x^2 - 8x + 20 = 0$ are 2 and $-\frac{5}{3}$; what are the others?

We first put the equation in the general form by dividing each term by 9.

It then becomes $x^4 - \frac{37}{9}x^2 - \frac{8}{9}x + \frac{20}{9} = 0.$

Since there is no x^3 term, the coefficient of the second term is 0.

Then the coefficient of the second term of the depressed equation is $0 + 2 - \frac{5}{3}$ or $\frac{1}{3}$.

The coefficient of the last term of the depressed equation is

$$\frac{20}{9} \div \left(-\frac{10}{3}\right), \text{ or } -\frac{2}{3}.$$

Then, the depressed equation is $x^2 + \frac{1}{3}x - \frac{2}{3} = 0.$

Solving, $x = \frac{2}{3}$ or $-1.$

EXERCISE 78

1. One root of $x^3 + 7x^2 - 5x - 75 = 0$ is -5; find the others.

2. Three roots of $4x^4 - 55x^2 - 45x + 36 = 0$ are 4, -3, $\frac{1}{2}$; find the other.

3. Four roots of $20x^5 - 108x^4 + 225x^3 - 224x^2 + 105x - 18 = 0$ are 1, 1, $\frac{2}{5}$, $\frac{3}{2}$; find the other.

4. Three roots of $x^5 - x^4 - 15x^3 + 25x^2 + 14x - 24 = 0$ are 2, 1, -4; find the others.

5. Two roots of $x^4 - ax^3 + (2a - 7a^2 - 1)x^2 + (a^3 - 2a^2 + a)x + 6a^4 - 12a^3 + 6a^2 = 0$ are $a - 1$ and $3a$; find the others.

322. Fractional Roots.

An equation in the general form with integral coefficients cannot have as a root a rational fraction in its lowest terms.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$

where $p_1, p_2, \cdots, p_n$ are integral.

If possible, let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of the equation; then,

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0.$$

Multiplying each term by b^{n-1}, and transposing,

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}).$$

By hypothesis, a and b have no common divisor; hence, a^n and b have no common divisor.

We then have a rational fraction in its lowest terms equal to an integral expression, which is impossible.

Therefore, the equation cannot have as a root a rational fraction in its lowest terms.

323. Imaginary Roots.

If the imaginary number $a + bi$ is a root of an equation in the general form, with real coefficients, its conjugate $(a - bi)$ is also a root.

Let the equation be

$$x^n + p_1 x^{n-1} + \cdots + p_{n-1}x + p_n = 0, \tag{1}$$

where $p_1, \cdots, p_n$ are real numbers.

Since $a + bi$ is a root of (1), we have

$$(a + bi)^n + p_1(a + bi)^{n-1} + \cdots + p_{n-1}(a + bi) + p_n = 0.$$

Expanding by the Binomial Theorem, we have, by § 180,

$$a^n + na^{n-1}bi - \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 - \frac{n(n-1)(n-2)}{\lfloor 3} a^{n-3}b^3 i + \cdots$$

$$+ p_1\left[a^{n-1} + (n-1)a^{-2}bi - \frac{(n-1)(n-2)}{\lfloor 2} a^{n-3}b^2 - \cdots\right]$$

$$+ \cdots + p_{n-1}(a + bi) + p_n = 0. \tag{2}$$

Collecting the real and imaginary terms, we have a result of the form

$$P + Qi = 0. \tag{3}$$

Here, P stands for the sum of all the terms containing a alone, together with all the terms containing *even* powers of i; and Qi for all terms containing *odd* powers of i.

In order that equation (3) may hold, we must have

$$P = 0, \text{ and } Q = 0.$$

Now substituting $a - bi$ for x in the first member of equation (1), it becomes

$$(a - bi)^n + p_1(a - bi)^{n-1} + \cdots + p_{n-1}(a - bi) + p_n. \tag{4}$$

Expanding by the Binomial Theorem, we shall have a result which differs from the first member of (2) only in having the second, fourth, sixth, etc., terms of each expansion, or those involving i as a factor, *changed in sign.*

Then, collecting the real and imaginary terms, the expression (3) is equal to

$$P - Qi,$$

where P and Q have the same meanings as before.

But since $P = 0$ and $Q = 0$, we have $P - Qi = 0$.

Whence, $a - bi$ is a root of (1).

The above proof holds without change when a equals zero; thus the theorem holds for any pure imaginary number, of the form bi.

324. The product of the factors of the first member of equation (1), § 323, corresponding to the conjugate imaginary roots $a+bi$ and $a-bi$ is

$$[x-(a+bi)][x-(a-bi)] \qquad (\S\ 318)$$
$$=(x-a-bi)(x-a+bi)$$
$$=(x-a)^2-(bi)^2=(x-a)^2+b^2;$$

and is therefore positive for every real value of x.

325. It follows from §§ 316 and 323 that every equation of odd degree has at least one real root; for an equation cannot have an odd number of imaginary roots.

TRANSFORMATION OF EQUATIONS

326. *To transform an equation into another which shall have the same roots with contrary signs.*

Let the equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0. \qquad (1)$$

Substituting $-y$ for x, we have

$$(-y)^n+p_1(-y)^{n-1}+p_2(-y)^{n-2}+\cdots+p_{n-1}(-y)+p_n=0.$$

Dividing each term by $(-1)^n$, we have

$$y^n+p_1\frac{y^{n-1}}{(-1)}+p_2\frac{y^{n-2}}{(-1)^2}+\cdots+p_{n-1}\frac{y}{(-1)^{n-1}}+\frac{p^n}{(-1)^n}=0.$$

Or, $$y^n-p_1y^{n-1}+p_2y^{n-2}-\cdots\pm p_{n-1}y\mp p_n=0; \qquad (2)$$

the upper or lower signs being taken according as n is odd or even.

It follows from (1) and (2) that the desired transformation may be effected by simply **changing the signs of the alternate terms commencing with the second.**

If the equation is *incomplete*, any missing term must be supplied with the coefficient zero before applying the rule.

327. *Ex.* Transform the equation $x^3 - 10x + 4 = 0$ into another which shall have the same roots with contrary signs.

The equation may be written $x^3 + 0 \cdot x^2 - 10x + 4 = 0$.
Then, by the rule, the transformed equation is

$$x^3 - 0 \cdot x^2 - 10x - 4 = 0, \text{ or } x^3 - 10x - 4 = 0.$$

EXERCISE 79

Transform each of the following into an equation which shall have the same roots with contrary signs:

1. $x^3 - 6x^2 + 12x - 8 = 0$.
2. $x^4 - 6x^3 + 4x^2 - 9x + 16 = 0$.
3. $x^7 + 5x^5 - 3x^4 + x - 4 = 0$.

328. *To transform an equation into another whose roots shall be respectively m times those of the first.*

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Putting $mx = y$, that is, $\frac{y}{m}$ for x, we have

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{y}{m}\right) + p_n = 0.$$

Multiplying each term by m^n,

$$y^n + p_1my^{n-1} + p_2m^2y^{n-2} + \cdots + p_{n-1}m^{n-1}y + p_nm^n = 0.$$

Hence, to effect the desired transformation, **multiply the second term by m, the third term by m², and so on.**

Ex. Transform the equation $x^3 + 7x^2 - 6 = 0$ into another whose roots shall be respectively 4 times those of the first.

Supplying the missing term with the coefficient zero, and applying the rule, we have

$$x^3 + 4 \cdot 7x^2 + 4^2 \cdot 0x - 4^3 \cdot 6 = 0, \text{ or } x^3 + 28x^2 - 384 = 0.$$

329. *To transform an equation with fractional coefficients into another whose coefficients shall be integral, that of the first term being unity.*

The transformation may be effected by transforming the equation into another whose roots shall be respectively m times those of the first (§ 328); we then give to m such a value as will make every coefficient integral.

By giving to m the *least* value which will make every coefficient integral, the result will be obtained in its simplest form.

Ex. Transform the equation $x^3 - \frac{x^2}{3} - \frac{x}{36} - \frac{1}{108} = 0$ into another whose coefficients shall be integral, that of the first term being unity.

By § 328, the equation

$$x^3 - \frac{m}{3}x^2 - \frac{m^2}{36}x + \frac{m^3}{108} = 0$$

has its roots respectively m times those of the given equation.

It is evident, by inspection, that the least value of m which will make every coefficient integral, is 6.

Putting $m = 6$, we have

$$x^3 - 2x^2 - x + 2 = 0,$$

whose roots are 6 times those of the given equation.

330. *To transform an equation into another whose roots shall be respectively those of the first increased by m.*

Let the equation be

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

Putting $x + m = y$, that is, $y - m$ for x, we have

$$(y-m)^n + p_1(y-m)^{n-1} + \cdots + p_{n-1}(y-m) + p_n = 0. \qquad (2)$$

Expanding the powers of $y - m$ by the Binomial Theorem, and collecting the terms involving like powers of y, we shall have a result of the form

$$y^n + q_1y^{n-1} + \cdots + q_{n-1}y + q_n = 0, \qquad (3)$$

whose roots are respectively those of the given equation increased by m.

Ex. Transform the equation $x^3 - 7x + 6 = 0$ into another whose roots shall be respectively those of the first increased by 2.

Substituting $y - 2$ for x,

$$(y-2)^3 - 7(y-2) + 6 = 0.$$

Expanding, and collecting the terms involving like powers of y, we have

$$y^3 - 6y^2 + 5y + 12 = 0.$$

331. If m and the coefficients of the given equation are integral, the coefficients of the transformed equation may be conveniently found by the following method.

Putting $x + m$ for y in (3), we obtain

$$(x+m)^n + q_1(x+m)^{n-1} + \cdots + q_{n-1}(x+m) + q_n = 0, \qquad (4)$$

which must, of course, take the same form as (1) on expanding the powers of $x + m$, and collecting the terms involving like powers of x.

Dividing the first member of (4) by $x + m$, we have

$$(x+m)^{n-1} + q_1(x+m)^{n-2} + \cdots + q_{n-2}(x+m) + q_{n-1} \qquad (5)$$

as a quotient, with a remainder q_n.

Dividing (5) by $x + m$, we have the remainder q_{n-1}; etc.

Hence, to obtain the coefficients of the transformed equation:

Divide the first member of the given equation by $x + m$; the remainder will be the last term of the required equation.

Divide the quotient just found by $x + m$; the remainder will be the coefficient of the next to the last term of the transformed equation; and so on.

Ex. Transform the equation $x^3 - 7x + 6 = 0$ into another whose roots shall be respectively those of the first increased by 2.

Dividing $x^3 - 7x + 6$ by $x + 2$, we have the quotient $x^2 - 2x - 3$, and the remainder 12 (§ 108).

Dividing $x^2 - 2x - 3$ by $x + 2$, we have the quotient $x - 4$, and the remainder 5.

Dividing $x - 4$ by $x + 2$, we have the remainder -6.

Then, the transformed equation is

$$x^3 - 6x^2 + 5x + 12 = 0.$$

Compare Ex., § 330.

332. To transform an equation into another whose roots shall be those of the first *diminished by m*, we change $y-m$ to $y+m$ in the method of § 330, and $x+m$ to $x-m$ in the rule of § 331.

EXERCISE 80

1. Transform $x^2-x-12=0$ into an equation whose roots shall be, respectively, 5 times the first. Verify your results.

2. Transform $x^3+x^2-14x-24=0$ into an equation whose roots shall be, respectively, twice those of the first. Verify results.

3. Transform $x^3+8x^2-23x-210=0$ into an equation whose roots shall be, respectively, $\frac{1}{2}$ times the first.

Transform each of the following into an equation with integral coefficients, that of the first term being unity:

4. $6x^3-11x^2-14x+24=0$. Verify result.

5. $8x^3+14x^2-5x-2=0$.

6. $2x^4-13x^3-91x^2+390x+216=0$.

7. $90x^4+111x^3+25x^2-12x-4=0$.

8. $x^4+\frac{8x^3}{7}-\frac{5x^2}{14}-\frac{1}{196}=0$.

9. Transform $x^3+10x^2+7x-18=0$ into an equation whose roots shall be, respectively, those of the first diminished by 4.

10. Transform $x^4-3x^3-19x^2+27x+90=0$ into an equation whose roots shall be, respectively, those of the first increased by 3.

333. *To transform the equation*

$$x^n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n=0$$

where p_1 is not zero, into another whose second term shall be wanting.

Expanding the powers of $y-m$ in the first member of (2), § 330, and collecting the terms involving like powers of y, we have

$$y^n + (p_1 - mn)y^{n-1} + \cdots = 0.$$

If m be so taken that $p_1 - mn = 0$, whence $m = \frac{p_1}{n}$, the coefficient of y^{n-1} will be zero.

Hence, the desired transformation may be effected by substituting in the given equation $y - \frac{p_1}{n}$ in place of x.

Ex. Transform $x^3 - 6x^2 + 9x - 6 = 0$ into an equation whose second term shall be wanting.

Substituting $y - \frac{-6}{3}$ or $y + 2$, in place of x, we have

$$(y+2)^3 - 6(y+2)^2 + 9(y+2) - 6 = 0.$$

Then, $y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 9y + 18 - 6 = 0,$

or $$y^3 - 3y - 4 = 0;$$

whose roots are those of the given equation diminished by 2.

EXERCISE 81

Transform each of the following into an equation whose second term shall be wanting:

1. $x^3 - 6x^2 + 4x - 1 = 0.$
2. $x^3 + 5x^2 + 8 = 0.$
3. $x^4 + 12x^3 + 2x^2 - 3 = 0.$
4. $x^5 - x^4 + 7x - 1 = 0.$

DESCARTES' RULE OF SIGNS

334. If an equation of the nth degree is in the general form (§ 312), a *Permanence* of sign occurs when two successive terms have the *same* sign, and a *Variation* of sign occurs when two successive terms have *opposite* signs.

Thus, in the equation $x^6 - 3x^4 - x^3 + 5x + 1 = 0$, there are two permanences and two variations.

335. Descartes' Rule of Signs.

No equation, whether complete or incomplete, can have a greater number of positive roots than it has variations of sign; and no complete equation can have a greater number of negative roots than it has permanences of sign.

Let an equation in the general form have the following signs:

$$+ + 0 - + 0 0 - -,$$

the missing terms being supplied with zero coefficients.

If we introduce a new positive root a, we multiply this by $x - a$ (§ 318); writing only the signs which occur in the process, we have

$$\begin{array}{cccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & \\
+ & + & 0 & \dot{-} & \dot{+} & 0 & 0 & \dot{-} & - & & (1) \\
+ & - & & & & & & & & & \\
\hline
+ & + & 0 & - & + & 0 & 0 & - & - & & \\
 & - & - & 0 & + & - & 0 & 0 & + & + & \\
\hline
+ & m & - & - & + & - & 0 & - & m & + & (2) \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 &
\end{array}$$

Here m signifies a term which may be $+$, 0, or $-$.

Now, in (1), let a dot be placed over the first minus sign, then over the next plus sign, then over the next minus sign, and so on.

The number of dots shows the number of variations; thus in (1) there are three variations.

In the above result, we observe the following laws:

I. Directly under each dotted term of (1) is a term of (2) *having the same sign.*

Thus, the terms numbered 4, 5, and 8, in (1) and (2), have the same sign.

II. The last term of (2) is of opposite sign to the term directly under the last dotted term of (1).

The above laws are easily seen to hold universally.

By the first law, however the term marked m is taken, there are at least as many variations in the first eight terms of (2) as in (1); and by the second law, there is at least one variation in the remaining terms of (2).

Hence, the introduction of a new positive root increases the number of variations in the equation by at least one.

If, then, we form the product of all the factors corresponding to the negative and imaginary roots of an equation, multiplying the result by the factor corresponding to each positive root introduces at least one variation.

Hence, the equation cannot have a greater number of positive roots than it has variations of sign.

To prove the second part of Descartes' Rule, let $-y$ be substituted for x in any *complete* equation.

Then since the signs of the alternate terms commencing with the second are changed (§ 326), the original *permanences* of sign become *variations*.

But the transformed equation cannot have a greater number of *positive* roots than it has *variations*.

Hence, the original equation cannot have a greater number of *negative* roots than it has *permanences*.

In all applications of Descartes' Rule, the equation must contain a term independent of x; that is, no root must equal zero; for a zero root cannot be regarded as either positive or negative.

336. It follows from the last part of § 335, and from § 326, that in any equation, whether complete or incomplete, the number of negative roots cannot exceed the number of variations in the equation which is formed from the given equation by changing the signs of the alternate terms commencing with the second.

337. In any *complete* equation, the sum of the number of permanences and variations is equal to the number of terms less one, or to the degree of the equation.

That is, the sum of the number of permanences and variations is equal to the number of roots (§ 316).

Hence, if the roots of a complete equation are all real, the number of positive roots equals the number of variations, and the number of negative roots equals the number of permanences.

An equation whose terms are all positive can have no positive root; and a complete equation whose terms are alternately positive and negative can have no negative root.

338. *Ex.* Determine the nature of the roots of

$$x^3 + 2x + 5 = 0.$$

There is no variation, and consequently no positive root.

Changing the signs of the alternate terms commencing with the second, we have $x^3 + 2x - 5 = 0$. (See Note, § 326.)

Here there is one variation; and therefore the given equation cannot have more than one negative root (§ 336).

Then since the equation has three roots (§ 316), one of them must be negative and the other two imaginary.

If two or more successive terms of an equation are wanting, it follows by Descartes' Rule that the equation must have imaginary roots.

EXERCISE 82

If the roots of the following are all real, determine their signs:

1. $x^3 + 10x^2 + 7x - 18 = 0.$
2. $x^4 - 3x^3 - 19x^2 + 27x + 90 = 0.$
3. $36x^5 - 67x^3 + 27x^2 + 7x - 3 = 0.$
4. $x^5 - 4x^4 - 5x^3 + 20x^2 + 4x - 16 = 0.$
5. $2x^4 - 13x^3 - 91x^2 + 390x + 216 = 0.$

Determine the nature of the roots of the following:

6. $2x^3 + x^2 + 2x - 12 = 0.$
7. $x^4 + 3x^3 + 7x^2 + 6x + 4 = 0.$
8. $x^4 - 2x^3 - 9 = 0.$
9. $x^5 - 2x^4 + 4x^3 - 8x^2 + 16x - 16 = 0.$
10. $x^7 + 3x^4 + 5x^2 + 2 = 0.$

LIMITS TO THE ROOTS

339. *To find a superior limit to the positive roots of an equation.*

The following examples illustrate the method of finding a superior limit to the positive roots of an equation.

1. Find a superior limit to the positive roots of

$$x^3 - 3x^2 + 2x - 5 = 0.$$

Grouping the positive and negative terms, we can write the first member in the form

$$x^2(x-3) + 2\left(x - \tfrac{5}{2}\right). \qquad (1)$$

It is evident that if x equals or exceeds 3, the expression (1) is positive. Hence, no root of the given equation equals or exceeds 3, and 3 is a superior limit to the positive roots.

2. Find a superior limit to the positive roots of

$$x^4 - 15x^2 - 10x + 24 = 0.$$

We separate the first term into the parts $\frac{2x^4}{3}$ and $\frac{x^4}{3}$, and write the first member in the form

$$\left(\frac{2x^4}{3} - 15x^2\right) + \left(\frac{x^4}{3} - 10x\right) + 24, \text{ or } \frac{x^2}{3}(2x^2 - 45) + \frac{x}{3}(x^3 - 30) + 24.$$

It is evident from this that no root can be so great as 5; hence, 5 is a superior limit to the positive roots.

If we had written the first member in the form

$$\left(\frac{x^4}{2} - 15x^2\right) + \left(\frac{x^4}{2} - 10x\right) + 24, \text{ or } \frac{x^2}{2}(x^2 - 30) + \frac{x}{2}(x^3 - 20) + 24,$$

we should have found 6 as a superior limit to the positive roots.

Thus, separating x^4 into $\frac{2x^4}{3}$ and $\frac{x^4}{3}$, instead of $\frac{x^4}{2}$ and $\frac{x^4}{2}$, gives a smaller limit.

340. *To find an inferior limit to the negative roots of an equation.*

First transform the equation into another which shall have the same roots with contrary signs (§ 326).

The superior limit to the positive roots of the transformed equation, obtained as in § 339, with its sign changed, will be an inferior limit to the negative roots of the given equation.

Ex. Find an inferior limit to the negative roots of

$$x^5 + 2x^3 + 5x^2 - 7 = 0.$$

Transforming the equation into another which shall have the same roots with contrary signs (§ 326), we have

$$x^5 + 2x^3 - 5x^2 + 7 = 0. \quad (1)$$

We can write the first member in the form

$$x^2(x^3 - 5) + 2x^3 + 7.$$

It is evident from this that no root of (1) can be so great as 2; hence, -2 is an inferior limit to the negative roots of the given equation.

By grouping the x^5 and x^2 terms in (1), we obtain a smaller limit than if we group the x^3 and x^2 terms.

EXERCISE 83

In each of the following, find a superior limit to the positive roots, and an inferior limit to the negative:

1. $x^3 + 3x^2 + x - 4 = 0.$
2. $x^4 + 5x^3 - 15x - 9 = 0.$
3. $x^4 + 3x^2 - 5x - 8 = 0.$
4. $3x^4 - 5x^2 - 8x - 7 = 0.$
5. $x^5 - 4x^4 + 6x^3 + 32x^2 - 15x + 3 = 0.$
6. $2x^5 + 5x^4 + 6x^3 - 13x^2 - 25x + 4 = 0.$
7. In the equation $x^3 - 2x^2 - 3x + 1 = 0$, prove 3 a superior limit to the positive roots, and -2 an inferior limit to the negative.
8. In the equation $2x^3 + 5x^2 - 7x - 3 = 0$, prove -4 an inferior limit to the negative roots, and find a superior limit to the positive.
9. In the equation $x^4 + 3x^3 - 9x^2 + 12x - 10 = 0$, prove 3 a superior limit to the positive roots, and -6 an inferior limit to the negative.

DERIVATIVES

341. If we take the polynomial

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots,$$

multiply each term by the exponent of x in that term, and then diminish the exponent by 1, the result

$$nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \cdots$$

is called the **first derivative** of the given polynomial.

The first derivative of the first derivative is called the *second derivative* of the given polynomial; the first derivative of the second derivative is called the *third derivative*; and so on.

Ex. Find the successive derivatives of $3x^3 - 9x^2 - 12x + 2$.

The first is $9x^2 - 18x - 12$.
The second is $18x - 18$.
The third is 18.
The fourth is 0.

It will be understood hereafter that when we speak of *the* derivative of an expression, we mean the *first derivative.*

EXERCISE 84

Find the successive derivatives of:

1. $5x^2 + 8x - 7$.
2. $3x^2 - 7x + 2$.
3. $9x^3 - 7x^2 + 15x - 1$.
4. $8x^5 - 3x^2 + 2$.
5. $6x^6 - 5x^5 + 4x^3 - 3x^2 + 27$.
6. $x^5 - x^4 + 10x^3 + 5x^2 - 7x$.

MULTIPLE ROOTS

342. If an equation has two or more roots equal to a, a is said to be a **Multiple Root** of the equation.

In the above case, a is called a *double root*, a *triple root*, a *quadruple root*, etc., according as the equation has two roots, three roots, four roots, etc., equal to a.

343. Let the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0 \qquad (1)$$

be a, b, c, d, $\cdots$.

Then, by § 318, we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots = (x-a)(x-b)(x-c)\cdots.$$

Putting $x + h$ in place of x, we obtain

$$(x+h)^n + p_1(x+h)^{n-1} + p_2(x+h)^{n-2} + \cdots$$
$$= (h + \overline{x-a})(h + \overline{x-b})(h + \overline{x-c})\cdots. \qquad (2)$$

Expanding the powers of $x + h$ by the Binomial Theorem, the coefficient of h in the first member of (2) is

$$nx^{n-1} + p_1(n-1)x^{n-2} + p_2(n-2)x^{n-3} + \cdots; \qquad (3)$$

which, we observe, is the first derivative of the first member of (1).

Again, it is evident from § 319 that the coefficient of h in the second member of (2) is

$$(x-b)(x-c)(x-d)\cdots \text{ to } n-1 \text{ factors}$$
$$+ (x-a)(x-c)(x-d)\cdots \text{ to } n-1 \text{ factors}$$
$$+ (x-a)(x-b)(x-d)\cdots \text{ to } n-1 \text{ factors} + \cdots. \qquad (4)$$

Since equation (2) is true for every value of h, by § 264 these coefficients of h in the two members are equal.

Now if $b = a$, that is, if equation (1) has *two* roots equal to a, every term of (4) will be divisible by $x - a$, and therefore the expression (3) will be divisible by $x - a$.

Hence, the equation formed by equating (3) to zero will have *one* root equal to a (§ 315).

In like manner, if $c = b = a$, that is, if (1) has *three* roots equal to a, the equation formed by equating (3) to zero will have *two* roots equal to a; and so on.

Hence, **if any equation of the form** (1) **has m roots equal to a, the equation formed by equating to zero the derivative of its first member will have $m - 1$ roots equal to a.**

344. It follows from § 343 that, to determine the existence of multiple roots in an equation of the form

$$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

we proceed as follows:

Find the H. C. F. of the first member and its derivative.

If there is no H. C. F., there can be no multiple roots.

If there is a H. C. F., by equating it to zero and solving the resulting equation, the required roots may be obtained.

It is to be observed that the number of times that each root occurs in the given equation exceeds by one the number of times that it occurs in the equation formed by equating the H. C. F. to zero.

Ex. Find all the roots of

$$x^5 + x^4 - 9x^3 - 5x^2 + 16x + 12 = 0. \quad (1)$$

The derivative of the first member is

$$5x^4 + 4x^3 - 27x^2 - 10x + 16.$$

The H. C. F. of this and the first member of (1) is $x^2 - x - 2$.

Solving the equation $x^2 - x - 2 = 0$, $x = 2$ or -1.

Then, the multiple roots of (1) are 2, 2, -1, and -1.

Subtracting the sum of 2, 2, -1, and -1 from -1, the remaining root is -3 (§ 321).

EXERCISE 85

Find all the roots of the following:

1. $x^3 - x^2 - 21x + 45 = 0$.
2. $x^4 + 6x^3 - 11x^2 - 60x + 100 = 0$.
3. $9x^3 + 105x^2 + 343x + 343 = 0$.
4. $4x^4 + 32x^3 + 63x^2 - 8x - 16 = 0$.
5. $x^5 + 5x^4 - 17x^3 - 49x^2 + 160x - 100 = 0$.
6. $x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$.

345. The equation $x^n - a = 0$ can have no multiple roots; for the derivative of $x^n - a$ is nx^{n-1}, and $x^n - a$ and nx^{n-1} have no common factor except unity.

Hence, the n roots of $x^n = a$ are all different.

It follows from this that every expression has two different square roots, three different cube roots, and, in general, n different nth roots.

LOCATION OF ROOTS

346. If two real numbers, *a* and *b*, not roots of the equation

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0, \qquad (1)$$

when substituted for *x* in the first member, give results of opposite sign, an odd number of roots of the equation lie between *a* and *b*.

Let a be *algebraically* greater than b.

Let $d, \cdots, g$ be the real roots of (1) lying between a and b, and $h, \cdots, k$, the remaining real roots.

Let $x_n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$ be denoted by X.

Then, by § 318,

$$X = (x-d)\cdots(x-g)\cdot(x-h)\cdots(x-k)\cdot Y; \qquad (2)$$

where Y denotes the product of the factors corresponding to the imaginary roots, if any, of (1).

Substituting a, and then b, for x in (2), the second member becomes

$$(a-d)\cdots(a-g)\cdot(a-h)\cdots(a-k)\cdot Y', \qquad (3)$$

and

$$(b-d)\cdots(b-g)\cdot(b-h)\cdots(b-k)\cdot Y''; \qquad (4)$$

where Y' and Y'' denote the values of Y when x is put equal to a and b, respectively.

Since a is greater than b, each of the numbers $d, \cdots, g$ is less than a and greater than b.

Whence, each of the factors $a-d, \cdots, a-g$ is $+$, and each of the factors $b-d, \cdots, b-g$ is $-$.

Again, since none of the numbers $h, \cdots, k$ lie between a and b, the expression $(a-h)\cdots(a-k)$ has the same sign as the expression

$$(b-h)\cdots(b-k).$$

Also, Y' and Y'' are positive; for the product of the factors corresponding to a pair of conjugate imaginary roots of (1) is positive for every real value of x (§ 324).

But by hypothesis, the expressions (3) and (4) are of opposite sign.

Hence, the number of factors $b-d, \cdots, b-g$ must be *odd;* that is, an odd number of roots lie between a and b.

If the numbers substituted differ by unity, it is evident that the integral part of at least one root is known.

Ex. Locate the roots of $x^3 + x^2 - 6x - 7 = 0$.

By Descartes' Rule (§ 335), the equation cannot have more than one positive, nor more than two negative roots.

The values of the first member for the values 0, 1, 2, 3, -1, -2, and -3 of x are as follows:

$x = 0;\ -7.$ $x = 2;\ -7.$ $x = -1;\ -1.$ $x = -3;\ -7.$

$x = 1;\ -11.$ $x = 3;\ 11.$ $x = -2;\ 1.$

Since the sign of the first member is $-$ when $x = 2$, and $+$ when $x = 3$, one root lies between 2 and 3.

The others lie between -1 and -2, and -2 and -3, respectively.

In locating roots by the above method, first make trial of the numbers 0, 1, 2, etc., continuing the process until the number of positive roots determined is the same as has been previously indicated by Descartes' Rule.

Thus, in the above example, the equation cannot have more than one positive root; and when one has been found to lie between 2 and 3, there is no need of trying 4, or any greater positive number.

The work may sometimes be abridged by finding a superior limit to the positive roots, and an inferior limit to the negative roots of the given equation (§§ 339, 340), for no number need be tried which does not fall between these limits.

EXERCISE 86

Locate the roots of the following:

1. $x^3 + 4x^2 - 6 = 0$.
2. $x^3 - 7x^2 + 6x + 5 = 0$.
3. $x^3 + 3x^2 - 7x + 2 = 0$.
4. $x^3 + 4x^2 + x - 3 = 0$.
5. $x^4 + 3x^3 - 4x - 1 = 0$.
6. $x^4 + x^3 - 19x^2 - 17x + 1 = 0$.
7. $x^4 - 4x^3 + 6x - 2 = 0$.
8. $x^4 - 7x^2 + x + 4 = 0$.

9. Prove that the equation $x^4 - 5x^2 - 7x - 2 = 0$ has one root between 2 and 3, and at least one between 0 and -1.

10. Prove that the equation $x^4 - 3x^3 + x^2 - 3x - 4 = 0$ has one root between 0 and -1, and at least one between 3 and 4.

11. Prove that the equation $x^3 + 5x + 4 = 0$ has one root between 0 and -1.

347. The method of § 346 is not sufficient to deal with every problem in location of roots.

Let it be required, for example, to locate the roots of

$$x^3 + 3x^2 + 2x + 1 = 0.$$

By § 325, the equation has at least one real root.

By Descartes' Rule, it has no positive root.

Putting x equal to 0, -1, -2, -3, the corresponding values of the first member are 1, 1, 1, and -5, respectively.

Then, the equation has either one root or three roots between -2 and -3; but the methods already given are not sufficient to determine which.

Sturm's Theorem (§ 350) affords a method for determining completely the number and situation of the real roots of an equation.

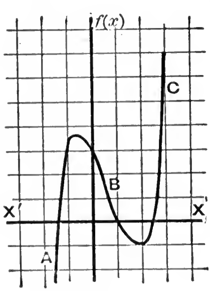
It is more difficult of application than the method of § 346, and should be used only in cases which the latter cannot resolve.

348. Graphical Representation.

The graph of an expression of higher degree than the second, with one unknown number, may be found as in § 51.

Ex. Find the graph of

$$x^3 - 2x^2 - 2x + 3.$$



Put $f(x) = x^3 - 2x^2 - 2x + 3$.

If $x = 0$, $f(x) = 3$.
If $x = 1$, $f(x) = 0$.
If $x = 2$, $f(x) = -1$.
If $x = -2$, $f(x) = -9$.
If $x = -1$, $f(x) = 2$.
If $x = 3$, $f(x) = 6$.
etc.

The graph is the curve ABC, which extends in either direction to an indefinitely great distance from XX'.

349. Graphical Location of Roots.

The principle of § 220 holds for the graph of the first member of an equation of higher degree than the second, with one unknown number.

Thus, the graph of § 348 intersects XX' at $x=1$, between $x=2$ and $x=3$, and between $x=-1$ and $x=-2$.

And the equation $x^3-2x^2-2x+3=0$ has one root equal to 1, one between 2 and 3, and one between -1 and -2.

This may be verified by solving the equation; the factors of the first member are $x-1$ and x^2-x-3.

This method of locating roots is simply a graphical representation of the process of § 346, and is subject to the limitations stated in § 347.

If the graph is tangent to XX', the equation has two or more equal roots (compare § 220, Fig. 2); if it does not intersect XX', the equation has no real root.

The note to the example of § 346 applies with equal force to the graphical method of locating roots.

EXERCISE 87

Locate the roots of the following graphically:

1. $x^3-3x-1=0$.
2. $x^4+2x^2+3=0$.
3. $x^3-7x^2+12x-5=0$.
4. $x^3-8x^2+19x-12=0$.
5. $x^3+7x^2+14x+8=0$.
6. $x^3-3x^2-2x+5=0$.

350. Sturm's Theorem.

Let
$$x^n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n=0 \qquad (1)$$
be an equation from which the multiple roots have been removed (§ 343).

Let $x^n+p_1x^{n-1}+\cdots+p^{n-1}x+p_n$ be denoted by $f(x)$, and let $f_1(x)$ denote the first derivative of $f(x)$ (§ 341).

Dividing $f(x)$ by $f_1(x)$, we shall obtain a quotient Q_1, with a remainder of a degree lower than that of $f_1(x)$.

Denote this remainder, *with the sign of each of its terms* *changed*, by $f_2(x)$, and divide $f_1(x)$ by $f_2(x)$, and so on; the operation being precisely the same as that of finding the H. C. F. of $f(x)$ and $f_1(x)$, except that the signs of the terms of each remainder are to be changed, while no other changes of sign are permissible.

Since, by hypothesis, $f(x)=0$ has no multiple roots, $f(x)$ and $f_1(x)$ have no common divisor except 1 (§ 343); and we shall finally obtain a remainder $f_n(x)$ independent of x.

The expressions $f(x), f_1(x), f_2(x), \cdots, f_n(x)$, are called *Sturm's Functions.*

The successive operations are represented as follows:

$$f(x) = Q_1 f_1(x) - f_2(x), \tag{2}$$

$$f_1(x) = Q_2 f_2(x) - f_3(x), \tag{3}$$

$$f_2(x) = Q_3 f_3(x) - f_4(x), \tag{4}$$

. . . .

$$f_{n-2}(x) = Q_{n-1} f_{n-1}(x) - f_n(x).$$

We may now enunciate Sturm's Theorem:

Let two real numbers, a and b, be substituted in place of x in Sturm's Functions, and the signs noted.

The difference between the number of variations of sign (§ 334) in the first case and that in the second is equal to the number of real roots of $f(x)=0$ lying between a and b.

The proof of the theorem depends upon the following principles:

I. *Two consecutive functions cannot both become* 0 *for the same value of x.*

For if, for any value of x, $f_1(x)=0$ and $f_2(x)=0$, then by (3), $f_3(x)=0$; and since $f_2(x)=0$ and $f_3(x)=0$, by (4) $f_4(x)=0$; continuing in this way, we shall finally have $f_n(x)=0$.

But by hypothesis, $f_n(x)$ is independent of x, and consequently cannot become 0 for any value of x.

Hence, no two consecutive functions can become 0 for the same value of x.

II. *If any function, except $f(x)$ and $f_n(x)$, becomes 0 for any value of x, the adjacent functions have opposite signs for this value of x.*

For if, for any value of x, $f_2(x)=0$, then, by (3), we must have $f_1(x) = -f_3(x)$ for this value of x.

Therefore, $f_1(x)$ and $f_3(x)$ must have opposite signs for this value of x; for, by I, neither of them can equal zero.

III. Let c be a root of the equation $f_r(x)=0$, where $f_r(x)$ is any function except $f(x)$ and $f_n(x)$.

By II, $f_{r-1}(x)$ and $f_{r+1}(x)$ have opposite signs when $x=c$.

Let h be a positive number, so taken that no root of $f_{r-1}(x) = 0$, or $f_{r+1}(x)=0$ lies between $c-h$ and $c+h$.

Then, as x changes from $c-h$ to $c+h$, no change of sign takes place in $f_{r-1}(x)$, or $f_{r+1}(x)$; while $f_r(x)$ reduces to zero, and changes or retains its sign according as the root c occurs an odd or even number of times in $f_r(x)=0$.

Therefore, for values of x between $c-h$ and c, and also for values of x between c and $c+h$, the three functions $f_{r-1}(x)$, $f_r(x)$, and $f_{r+1}(x)$ present one permanence and one variation.

Hence, as x increases from $c-h$ to $c+h$, no change occurs in the *number* of variations in the functions $f_{r-1}(x)$, $f_r(x)$, and $f_{r+1}(x)$; that is, no change occurs in the number of variations as x increases through a root of $f_r(x)=0$.

IV. Let c be a root of the equation $f(x)=0$; and let h be a positive number so taken that no root of $f_1(x)=0$ lies between $c-h$ and $c+h$.

Then as x increases from $c-h$ to $c+h$, no change of sign takes place in $f_1(x)$, while $f(x)$ reduces to zero, and changes sign.

Now if we put $x=c-h$ in (1), the first member becomes

$$(c-h)^n + p_1(c-h)^{n-1} + \cdots + p_{n-1}(c-h) + p_n.$$

Expanding the powers of $c-h$ by the Binomial Theorem, and collecting the terms involving like powers of h, we have

$$\begin{aligned} &c^n + p_1c^{n-1} + \cdots + p_{n-1}c + p_n \\ &\quad - h[nc^{n-1} + (n-1)p_1c^{n-2} + \cdots + p_{n-1}] \\ &\quad + \text{terms involving } h^2, h^3, \cdots, h^n. \end{aligned} \tag{5}$$

But since c is a root of $f(x) = 0$, we have by (1),

$$c^n + p_1c^{n-1} + \cdots + p_{n-1}c + p_n = 0.$$

Also, it is evident that the coefficient of $-h$ is the value of $f_1(x)$ when c is substituted in place of x; let this be denoted by A; then (5) reduces to

$$-hA + \text{terms involving } h^2, h^3, \cdots, h^n. \tag{6}$$

In like manner, the value of $f(x)$ when x is put equal to $c + h$, is

$$+hA + \text{terms involving } h^2, h^3, \cdots h^n. \tag{7}$$

Now, if h be taken sufficiently small, the signs of the expressions (6) and (7) will be the same as the signs of their first terms, $-hA$ and $+hA$, respectively.

Hence, if h be taken sufficiently small, the sign of (6) will be contrary to the sign of A, and the sign of (7) will be the same as the sign of A.

Therefore, for values of x between $c - h$ and c, the functions $f(x)$ and $f_1(x)$ present a variation, and for values of x between c and $c + h$ they present a permanence.

Hence, a variation is lost as x increases through a root of the equation $f(x) = 0$.

We may now prove Sturm's Theorem; for as x increases from b to a, supposing a *algebraically* greater than b, a variation is lost each time that x passes through a root of $f(x) = 0$, and only then; for when x passes through a root of $f_r(x) = 0$, where $f_r(x)$ is any function except $f(x)$ and $f_n(x)$, no change occurs in the number of variations.

Hence, the number of variations lost as x increases from b to a is equal to the number of real roots of $X = 0$ included between a and b.

351. It is customary, in applications of Sturm's Theorem, to speak of the substitution of an indefinitely great positive number for x, in an expression, as *substituting* $+\infty$ *for* x; and the substitution of a negative number of indefinitely great absolute value as *substituting* $-\infty$ for x.

The substitution of $+\infty$ and $-\infty$ for x in Sturm's Functions determines the number of real roots of $f(x)=0$.

The substitution of $+\infty$ and 0 for x determines the number of positive real roots, and the substitution of $-\infty$ and 0 the number of negative real roots.

Since Sturm's Theorem determines the number of real roots of an equation, the number of imaginary roots also becomes known (§ 316).

352. If a sufficiently great number be substituted in place of x in the expression

$$f(x)=p_0x^n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n,$$

the sign of the result will be the same as the sign of its first term, p_0x^n.

It follows from the above that:

If $+\infty$ be substituted in place of x in $f(x)$, the sign of the result will be the same as the sign of its first term.

If $-\infty$ be substituted in place of x in $f(x)$, the sign of the result will be the same as, or contrary to, the sign of the first term, according as the degree of $f(x)$ is even or odd.

353. Examples.

1. Determine the number and situation of the real roots of

$$x^3-2x^2-x+1=0.$$

Here, $f(x)=x^3-2x^2-x+1$, and $f_1(x)=3x^2-4x-1$.

In the process of finding $f_2(x)$, $f_3(x)$, etc., any positive numerical factors may be omitted or introduced at pleasure, for the *sign* of the result is not affected thereby; in this way fractions may be avoided.

In the present case, we multiply $f(x)$ by 3, to make its first term divisible by $3x^2$.

$$
\begin{array}{r}
3x^2-4x-1)\,3x^3-6x^2-3x+3\,(x \\
3x^3-4x^2-x \\
\hline
-2x^2-2x+3 \\
3 \\
\hline
-6x^2-6x+9\,(-2 \\
-6x^2+8x+2 \\
\hline
7)\,-14x+7 \\
\hline
-2x+1
\end{array}
\qquad \text{Then, } f_2(x)=2x-1.
$$

$$
\begin{array}{r}
3x^2-4x-1 \\
2 \\
\hline
2x-1)\,6x^2-8x-2\,(3x \\
6x^2-3x \\
\hline
-5x-2 \\
2 \\
\hline
-10x-4\,(-5 \\
-10x+5 \\
\hline
-9
\end{array}
\qquad \text{Then, } f_3(x)=9.
$$

Substituting $-\infty$ for x in $f(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$, the signs are $-$, $+$, $-$, and $+$, respectively (§ 352); substituting 0 for x, the signs are $+$, $-$, $-$, $+$, respectively; and substituting $+\infty$ for x, the signs are all $+$.

Hence, the roots of the equation are all real, and two of them are positive and the other negative (§ 351).

We now substitute various numbers to determine the situation of the roots:

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	
$x=-\infty$,	$-$	$+$	$-$	$+$	3 variations.
$x=-1$,	$-$	$+$	$-$	$+$	3 variations.
$x=0$,	$+$	$-$	$-$	$+$	2 variations.
$x=1$,	$-$	$-$	$+$	$+$	1 variation.
$x=2$,	$-$	$+$	$+$	$+$	1 variation.
$x=3$,	$+$	$+$	$+$	$+$	no variation.
$x=\infty$,	$+$	$+$	$+$	$+$	no variation.

We then know that the equation has one root between 0 and -1, one between 0 and 1, and one between 2 and 3.

2. Determine the number and situation of the real roots of

$$4x^3-6x-5=0.$$

Here, $f(x) = 4x^3 - 6x - 5$; and $f_1(x) = 12x^2 - 6$, or $2x^2 - 1$, omitting the factor 6.

$$\begin{array}{r|l}
2x^2 - 1) & 4x^3 - 6x - 5\,(2x \\
& \underline{4x^3 - 2x} \\
& \quad -4x - 5 \qquad \text{Then, } f_2(x) = 4x + 5.
\end{array}$$

$$\begin{array}{r|l}
& 2x^2 - 1 \\
& \underline{\qquad\quad 2} \\
4x + 5) & 4x^2 - 2\,(x \\
& \underline{4x^2 + 5x} \\
& \quad - 5x - 2 \\
& \underline{\qquad\qquad 4} \\
& -20x - 8\,(-5 \\
& \underline{-20x - 25} \\
& \qquad\quad 17 \qquad \text{Then, } f_3(x) = -17.
\end{array}$$

The last step in the division may be omitted; for we only need to know the *sign* of $f_3(x)$, and it is evident by inspection, when the remainder $-5x - 2$ is obtained, that the sign of $f_3(x)$ will be $-$.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$,	$-$	$+$	$-$	$-$	2 variations.
$x = 0$,	$-$	$-$	$+$	$-$	2 variations.
$x = 1$,	$-$	$+$	$+$	$-$	2 variations.
$x = 2$,	$+$	$+$	$+$	$-$	1 variation.
$x = \infty$,	$+$	$+$	$+$	$-$	1 variation.

Therefore, the equation has a real root between 1 and 2, and two imaginary roots.

In substituting the numbers, it is best to work from 0 in either direction, stopping when the number of variations is the same as has been previously found for $+\infty$ or $-\infty$, as the case may be.

EXERCISE 88

Determine the number and situation of the real roots of:

1. $x^3 + 2x^2 - x - 1 = 0.$
2. $x^3 + 3x - 5 = 0.$
3. $x^3 - 5x^2 + 2x + 6 = 0.$
4. $x^3 + x^2 - 15x - 28 = 0.$
5. $x^3 - 4x^2 + x + 3 = 0.$
6. $x^4 - 8x^2 - 8x + 1 = 0.$
7. $x^4 + 2x^3 - 5x^2 - 10x - 3 = 0.$
8. $x^4 + 3x^3 - 3x + 1 = 0.$

XV. SOLUTION OF HIGHER EQUATIONS

354. Synthetic Division (§ 107) not only abbreviates the process of division, but its application is of importance in the solution of many forms of higher equations containing either commensurable or incommensurable roots.

COMMENSURABLE ROOTS

355. We use the term *commensurable root*, in Chapter XV, to signify a *rational root* expressed in Arabic numerals.

356. By § 322, an equation of the nth degree in the general form (§ 312), with integral numerical coefficients, cannot have as a root a rational fraction in its lowest terms.

Therefore, to find all the commensurable roots of such an equation, we have only to find all its integral roots.

Again, by § 320, the last term of an equation of the above form is divisible by every integral root.

Hence, to find all the commensurable roots, we have only to *ascertain by trial which integral divisors of the last term are roots of the equation.*

The trial may be made in two ways:

I. By substitution of the supposed root.

II. By dividing the first member of the equation by the unknown number minus the supposed root (§ 315).

In this case, the operation may be conveniently performed by *Synthetic Division* (§ 107).

In the case of small numbers, such as ± 1, the first method may be the most convenient.

The second has the advantage that, when a root has been found, the process gives at once the depressed equation (§ 317) for obtaining the remaining roots.

Work may sometimes be saved by finding a superior limit to the positive, and an inferior limit to the negative, roots (§§ 339, 340); for no number need be tried which does not fall between these limits.

Descartes' Rule of Signs (§ 335) may also be advantageously employed to shorten the process.

Any multiple root should be removed (§ 343) before applying either method.

Ex. Find all the roots of $x^4 - 15x^2 + 10x + 24 = 0$.

By Descartes' Rule, the equation cannot have more than two positive roots.

Changing the signs of the alternate terms commencing with the second, we have $x^4 - 15x^2 - 10x + 24 = 0$.

Then, the given equation cannot have more than two negative roots (§ 336).

The integral divisors of 24 are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, and ± 24.

By substitution, we find that 1 is not, and that -1 is, a root of the equation.

Dividing the first member by $x - 2$, $x - 3$, etc., by the method explained in § 108, we have

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{2} \\ & 2 & 4 & -22 & -24 & \\ \hline & 2 & -11 & -12, & 0 & \text{Rem.} \end{array} \qquad \begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{3} \\ & 3 & 9 & -18 & -24 & \\ \hline & 3 & -6 & -8, & 0 & \text{Rem.} \end{array}$$

The work shows that 2 and 3 are roots of the given equation; and since the equation cannot have more than two positive roots, these are the only positive roots.

The remaining root may be found by dividing 24 by the product of -1, 2, and 3 (§ 321), or by the same process as above.

Dividing the first member by $x + 2$, $x + 3$, etc., we have

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-2} \\ & -2 & 4 & 22 & -64 & \\ \hline & -2 & -11 & 32 & -40 & \end{array} \qquad \begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-3} \\ & -3 & 9 & 18 & -84 & \\ \hline & -3 & -6 & 28 & -60 & \end{array}$$

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-4} \\ & -4 & 16 & -4 & -24 & \\ \hline & -4 & 1 & 6 & 0 & \end{array}$$

The work shows that the remaining root is -4.

357. By § 329, an equation of the nth degree in the general form, with fractional coefficients, may be transformed into another whose coefficients are integral, that of the first term being unity.

The commensurable roots of the transformed equation may then be found as in § 356.

Ex. Find all the roots of $4x^3 - 12x^2 + 27x - 19 = 0$.

Dividing through by the coefficient of x^3, we have

$$x^3 - 3x^2 + \frac{27x}{4} - \frac{19}{4} = 0.$$

Proceeding as in § 329, it is evident by inspection that the multiplier 2 will remove the fractional coefficients; the transformed equation is

$$x^3 - 2 \cdot 3x^2 + 2^2 \cdot \frac{27x}{4} - 2^3 \cdot \frac{19}{4} = 0,$$

or,
$$x^3 - 6x^2 + 27x - 38 = 0; \qquad (1)$$

whose roots are those of the given equation multiplied by 2.

By Descartes' Rule, equation (1) has no negative root.

The positive integral divisors of 38 are 1, 2, 19, and 38.

Dividing the first member by $x - 1$, $x - 2$, etc., we have

$$\begin{array}{rrrr|l} 1 & -6 & +27 & -38 & \underline{1} \\ & 1 & -5 & 22 & \\ \hline & -5 & 22 & -16 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -6 & +27 & -38 & \underline{2} \\ & 2 & -8 & 38 & \\ \hline & -4 & 19 & 0 & \end{array}$$

The work shows that 2 is a root of (1).

The remaining roots may now be found by depressing the equation; it is evident from the right-hand operation above that the depressed equation is $x^2 - 4x + 19 = 0$.

Solving by the rules for quadratics, $x = 2 \pm \sqrt{-15}$.

Then, the three roots of (1) are 2 and $2 \pm \sqrt{-15}$.

Dividing by 2, the roots of the given equation are 1 and $1 \pm \sqrt{-15}$.

EXERCISE 89

Find all the commensurable roots of the following, and the remaining roots when possible by methods already given.

1. $x^3 - 9x^2 + 23x - 15 = 0$.
2. $x^3 - 8x^2 + 5x + 14 = 0$.
3. $x^3 + 12x^2 + 44x + 48 = 0$.
4. $x^3 + 4x^2 - 9x - 36 = 0$.

5. $3x^3 + 4x^2 - 13x + 6 = 0.$

6. $4x^3 + 16x^2 - 7x - 39 = 0.$

7. $x^4 + 10x^3 + 35x^2 + 50x + 24 = 0.$

8. $x^4 - 5x^3 + 20x - 16 = 0.$

9. $x^4 - 15x^3 + 65x^2 - 105x + 54 = 0.$

10. $x^4 + 8x^3 + 11x^2 - 32x - 60 = 0.$

11. $x^4 - 2x^3 - 17x^2 + 18x + 72 = 0.$

12. $4x^4 - 12x^3 - 9x^2 + 47x - 30 = 0.$

13. $6x^4 - 7x^3 - 37x^2 + 8x + 12 = 0.$

14. $x^5 + 8x^4 - 7x^3 - 103x^2 + 69x + 18 = 0.$

15. $3x^4 + 2x^3 - 18x^2 + 8 = 0.$

16. $x^4 + x^3 - 6x^2 + 16x - 32 = 0.$

RECIPROCAL OR RECURRING EQUATIONS

358. A Reciprocal Equation is one such that if any number is a root of the equation, its reciprocal is also a root.

It follows from the above that, if $\frac{1}{x}$ be substituted for x in a reciprocal equation, the transformed equation will have the same roots as the given equation.

359. Let

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0 \qquad (1)$$

be a reciprocal equation.

Putting $\frac{1}{x}$ for x, the equation becomes

$$\frac{1}{x^n} + \frac{p_1}{x^{n-1}} + \frac{p_2}{x^{n-2}} + \cdots + \frac{p_{n-2}}{x^2} + \frac{p_{n-1}}{x} + p_n = 0.$$

Clearing of fractions, and reversing the order of the terms,

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + 1 = 0.$$

Dividing through by p_n,

$$x^n+\frac{p_{n-1}}{p_n}x^{n-1}+\frac{p_{n-2}}{p_n}x^{n-2}+\cdots+\frac{p_2}{p_n}x^2+\frac{p_1}{p_n}x+\frac{1}{p_n}=0. \qquad (2)$$

By § 358, this equation has the same roots as (1); and hence the following relations must hold between the coefficients of (1) and (2),

$$p_1=\frac{p_{n-1}}{p_n},\ p_2=\frac{p_{n-2}}{p_n},\ \cdots,\ p_{n-2}=\frac{p_2}{p_n},\ p_{n-1}=\frac{p_1}{p_n},\ p_n=\frac{1}{p_n}. \qquad (3)$$

From the last equation, $p_n^2=1$; whence, $p_n=\pm 1$.

Substituting the value of p_n, the equations (3) become

$$p_1=\pm p_{n-1},\ p_2=\pm p_{n-2},\ \cdots;$$

all the upper signs, or all the lower signs, being taken together.

We then have four varieties of reciprocal equations:

1. Degree odd, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *like* sign; as, $x^3-2x^2-2x+1=0$.

2. Degree odd, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *opposite* sign; as, $3x^5+2x^4-x^3+x^2-2x-3=0$.

3. Degree even, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *like* sign; as, $x^4-5x^3+6x^2-5x+1=0$.

4. Degree even, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *opposite* sign, and middle term wanting; as,

$$2x^6+3x^5-7x^4+7x^2-3x-2=0.$$

On account of the properties stated above, reciprocal equations are also called *Recurring Equations*.

360. Every reciprocal equation of the first variety may be written in the form

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_2x^2+p_1x+p_0=0,$$

or, $$p_0(x^n+1)+p_1(x^{n-1}+x)+p_2(x^{n-2}+x^2)+\cdots=0; \quad (1)$$

or, $$p_0(x^n+1)+p_1x(x^{n-2}+1)+p_2x^2(x^{n-4}+1)+\cdots=0;$$

the number of terms being even.

Since n is odd, each of the expressions x^n+1, $x^{n-2}+1$, etc., is divisible by $x+1$.

Therefore, -1 is a root of the equation (§ 315).

Dividing the first member of (1) by $x+1$, the depressed equation is

$$p_0(x^{n-1}-x^{n-2}+x^{n-3}-\cdots+x^2-x+1)$$
$$+p_1(x^{n-2}-x^{n-3}+x^{n-4}-\cdots+x^3-x^2+x)$$
$$+p_2(x^{n-3}-x^{n-4}+x^{n-5}-\cdots+x^4-x^3+x^2)+\cdots=0.$$

Or, $$p_0x^{n-1}+(p_1-p_0)x^{n-2}+(p_2-p_1+p_0)x^{n-3}+\cdots$$
$$+(p_2-p_1+p_0)x^2+(p_1-p_0)x+p_0=0;$$

which is a reciprocal equation of the third variety.

361. Every reciprocal equation of the second variety may be written in the form

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots-p_2x^2-p_1x-p_0=0,$$

or, $$p_0(x^n-1)+p_1(x^{n-1}-x)+p_2(x^{n-2}-x^2)+\cdots=0, \quad (1)$$

or, $$p_0(x^n-1)+p_1x(x^{n-2}-1)+p_2x^2(x^{n-4}-1)+\cdots=0.$$

Since each of the expressions x^n-1, $x^{n-2}-1$, etc., is divisible by $x-1$, $+1$ is a root of the equation.

Dividing the first member of (1) by $x-1$, the depressed equation is

$$p_0(x^{n-1}+x^{n-2}+x^{n-3}+\cdots+x^2+x+1)$$
$$+p_1(x^{n-2}+x^{n-3}+x^{n-4}+\cdots+x^3+x^2+x)$$
$$+p_2(x^{n-3}+x^{n-4}+x^{n-5}+\cdots+x^4+x^3+x^2)+\cdots=0,$$

or, $$p_0x^{n-1}+(p_1+p_0)x^{n-2}+(p_2+p_1+p_0)x^{n-3}+\cdots$$
$$+(p_2+p_1+p_0)x^2+(p+p_0)x+p_0=0;$$

which is a reciprocal equation of the third variety.

362. Every reciprocal equation of the fourth variety may be written in the form

$$p_0(x^n - 1) + p_1(x^{n-1} - x) + p_2(x^{n-2} - x^2) + \cdots = 0, \quad (1)$$

or, $$p_0(x^n - 1) + p_1x(x^{n-2} - 1) + p_2x^2(x^{n-4} - 1) + \cdots = 0;$$

the number of terms being even (§ 359).

Since each of the expressions $x^n - 1$, $x^{n-2} - 1$, etc., is divisible by $x^2 - 1$, both 1 and -1 are roots of the equation.

Dividing the first member of (1) by $x^2 - 1$, the depressed equation is

$$\begin{aligned} &p_0(x^{n-2} + x^{n-4} + x^{n-6} + \cdots + x^4 + x^2 + 1) \\ &+ p_1(x^{n-3} + x^{n-5} + x^{n-7} + \cdots + x^5 + x^3 + x) \\ &+ p_2(x^{n-4} + x^{n-6} + x^{n-8} + \cdots + x^6 + x^4 + x^2) + \cdots = 0, \end{aligned}$$

or, $$\begin{aligned} &p_0x^{n-2} + p_1x^{n-3} + (p_2 + p_0)x^{n-4} + \cdots \\ &+ (p_2 + p_0)x^2 + p_1x + p_0 = 0; \end{aligned}$$

which is a reciprocal of the third variety.

363. Every reciprocal equation of the third variety may be reduced to an equation of half its degree.

Let the equation be

$$\begin{aligned} &p_0x^{2m} + p_1x^{2m-1} + \cdots + p_{m-2}x^{m+2} + p_{m-1}x^{m+1} + p_mx^m \\ &+ p_{m-1}x^{m-1} + p_{m-2}x^{m-2} + \cdots + p_1x + p_0 = 0. \end{aligned}$$

Dividing each term by x^m, the equation may be written

$$\begin{aligned} &p_0\left(x^m + \frac{1}{x^m}\right) + p_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \cdots \\ &+ p_{m-2}\left(x^2 + \frac{1}{x^2}\right) + p_{m-1}\left(x + \frac{1}{x}\right) + p_m = 0. \quad (1) \end{aligned}$$

Put $x + \dfrac{1}{x} = y.$

Then, $$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2;$$

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) \\ &= y(y^2 - 2) - y = y^3 - 3y; \end{aligned}$$

$$x^4 + \frac{1}{x^4} = \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right) - \left(x^2 + \frac{1}{x^2}\right)$$

$$= y(y^3 - 3y) - (y^2 - 2) = y^4 - 4y^2 + 2\text{ ; etc.}$$

In general,

$$x^r + \frac{1}{x^r} = \left(x + \frac{1}{x}\right)\left(x^{r-1} + \frac{1}{x^{r-1}}\right) - \left(x^{r-2} + \frac{1}{x^{r-2}}\right);$$

an expression of the rth degree with respect to y.

Substituting these values in (1), the equation takes the form

$$q_0 y^m + q_1 y^{m-1} + q_2 y^{m-2} + \cdots = 0.$$

364. It follows from §§ 360 to 363 that any reciprocal equation of the degree $2m + 1$, and any reciprocal equation of the fourth variety of the degree $2m + 2$, can be reduced to an equation of the mth degree.

365. *Ex.* Solve $2x^5 - 5x^4 - 13x^3 + 13x^2 + 5x - 2 = 0$.

The equation being of the second variety, one root is 1 (§ 361).

Dividing by $x - 1$, the depressed equation is

$$2x^4 - 3x^3 - 16x^2 - 3x + 2 = 0;$$

a reciprocal equation of the third variety.

Dividing by x^2, $\quad 2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 16 = 0.$

Putting $x + \frac{1}{x} = y$, and $x^2 + \frac{1}{x^2} = y^2 - 2$ (§ 363), we have

$$2(y^2 - 2) - 3y - 16 = 0.$$

Solving this equation, $\quad y = 4 \text{ or } -\frac{5}{2}.$

Taking the first value, $\quad x + \frac{1}{x} = 4$, or $x^2 - 4x + 1 = 0.$

Whence, $\quad x = 2 \pm \sqrt{3}.$

Taking the second value, $x + \frac{1}{x} = -\frac{5}{2}$, or $2x^2 + 5x + 2 = 0.$

Whence, $\quad x = -2 \text{ or } -\frac{1}{2}.$

The roots of the given equation are $1, -2, -\frac{1}{2}$, and $2 \pm \sqrt{3}$.

That $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are reciprocals may be shown by multiplying them; thus, $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1.$

EXERCISE 90

Solve the following equations:

1. $4x^3 + 21x^2 + 21x + 4 = 0.$
2. $x^3 + 4x^2 - 4x - 1 = 0.$
3. $x^3 - 5x^2 - 5x + 1 = 0.$
4. $6x^4 + 13x^3 - 13x - 6 = 0.$
5. $24x^4 - 10x^3 - 77x^2 - 10x + 24 = 0.$
6. $x^5 + 2x^4 - 5x^3 + 5x^2 - 2x - 1 = 0.$
7. $5x^5 - 56x^4 + 131x^3 + 131x^2 - 56x + 5 = 0.$
8. $3x^5 + 4x^4 - 23x^3 - 23x^2 + 4x + 3 = 0.$
9. $6x^5 - 7x^4 - 27x^3 + 27x^2 + 7x - 6 = 0.$
10. $10x^6 - 19x^5 - 19x^4 + 19x^2 + 19x - 10 = 0.$

366. Binomial Equations.

A *Binomial Equation* is an equation of the form $x^n = a$.

Binomial equations are also reciprocal equations, and, in certain cases, may be solved by the method of § 365.

EXERCISE 91

Solve the following equations:

1. $x^5 = 1.$
2. $x^5 = -1.$
3. $x^5 = a^5.$ (Put $x = ay$.)

CUBIC EQUATIONS

367. A **Cubic Equation** is an equation of the *third degree* containing but one unknown number.

368. By § 333, the cubic equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0,$$

where p_1 is not zero, may be transformed into another whose second term shall be wanting by substituting $y - \frac{p_1}{3}$ for x.

Hence, every cubic equation can be reduced to the form

$$x^3 + ax + b = 0.$$

369. Cardan's Method for the Solution of Cubics.

Let it be required to solve the equation $x^3 + ax + b = 0$.

Putting $x = y + z$, the equation becomes

$$y^3 + 3\,yz(y + z) + z^3 + a(y + z) + b = 0,$$

or,
$$y^3 + z^3 + (3\,yz + a)(y + z) + b = 0.$$

We may give such a value to z that $3\,yz + a$ shall equal zero.

Whence,
$$z = -\frac{a}{3\,y}. \qquad (1)$$

Then,
$$y^3 + z^3 + b = 0. \qquad (2)$$

Substituting the value of z from (1) in (2), we have

$$y^3 - \frac{a^3}{27\,y^3} + b = 0, \text{ or } y^6 + by^3 - \frac{a^3}{27} = 0.$$

This is an equation in the quadratic form (Exercise 44, Note 3).

Solving by the rules for quadratics, we have

$$y^3 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}. \qquad (3)$$

Then by (2), $z^3 = -y^3 - b = -\frac{b}{2} \mp \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$. (4)

Taking the upper signs before the radical signs, in (3) and (4), and substituting in the equation $x = y + z$, we have

$$x = \sqrt[3]{\left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)} + \sqrt[3]{\left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)}. \qquad (5)$$

The *lower* signs before the radical signs give the *same value* of x.

The other two roots may be found by depressing the given equation (§ 317).

Ex. Solve the equation $x^3 + 3\,x^2 - 6\,x + 20 = 0$.

We first transform the equation into another whose second term shall be wanting.

Putting $x = y - \frac{3}{3} = y - 1$ (§ 368), we have

$$y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 - 6y + 6 + 20 = 0,$$

or

$$y^3 - 9y + 28 = 0.$$

To solve the latter equation, we substitute $a = -9$ and $b = 28$ in (5), § 369.

Thus,
$$y = \sqrt[3]{-14 + \sqrt{196 - 27}} + \sqrt[3]{-14 - \sqrt{196 - 27}}$$
$$= \sqrt[3]{-1} + \sqrt[3]{-27} = -1 - 3 = -4.$$

Therefore, $x = y - 1 = -5.$

Dividing the first member of the given equation by $x + 5$, the depressed equation is

$$x^2 - 2x + 4 = 0.$$

Solving,
$$x = 1 \pm \sqrt{-3}.$$

Thus, the roots of the given equation are -5 and $1 \pm \sqrt{-3}$.

EXERCISE 92

Solve the following equations:

1. $x^3 - 24x - 72 = 0.$
2. $x^3 - 12x + 16 = 0.$
3. $x^3 + 72x + 152 = 0.$
4. $x^3 - 12x^2 + 21x - 10 = 0.$
5. $x^3 - 3x^2 + 48x + 52 = 0.$
6. $x^3 + 6x^2 + 27x - 86 = 0.$
7. $x^3 + 9x^2 + 12x - 144 = 0.$
8. $x^3 + x^2 - 3x + 36 = 0.$
9. $x^3 - 2x^2 - 15x + 36 = 0.$
10. $x^3 - 4x^2 + 8x - 8 = 0.$
11. Find one root of $x^3 + x - 2 = 0.$

370. If a is negative, and $\frac{a^3}{27}$ numerically greater than $\frac{b^2}{4}$, the expression $\sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$ is imaginary.

In such a case, Cardan's method is of no practical value; for there is no method in Algebra for finding the cube root of a binomial surd.

In this case, which is called the *Irreducible Case*, Cardan's method is said to *fail*.

It is possible, in cases where Cardan's method fails, to find the roots by a method involving Trigonometry.

But practically it is easier to find them by the method of § 356, or by Horner's method (§ 374), according as the equation has or has not a commensurable root.

BIQUADRATIC EQUATIONS

371. A **Biquadratic Equation** is an equation of the *fourth degree,* containing but one unknown number.

372. Euler's Method for the Solution of Biquadratics.

By § 333, every biquadratic can be reduced to the form

$$x^4 + ax^2 + bx + c = 0. \quad (1)$$

Let $$x = u + y + z.$$

Then, $$x^2 = u^2 + y^2 + z^2 + 2\,uy + 2\,yz + 2\,zu,$$

or, $$x^2 - (u^2 + y^2 + z^2) = 2(uy + yz + zu).$$

Squaring both members, we have

$$x^4 - 2\,x^2(u^2 + y^2 + z^2) + (u^2 + y^2 + z^2)^2$$
$$= 4(u^2y^2 + y^2z^2 + z^2u^2 + 2\,uy^2z + 2\,u^2yz + 2\,yz^2u)$$
$$= 4(u^2y^2 + y^2z^2 + z^2u^2) + 8\,uyz(u + y + z).$$

Substituting x for $u + y + z$ and transposing,

$$x^4 - 2\,x^2(u^2 + y^2 + z^2) - 8\,uyzx$$
$$+ (u^2 + y^2 + z^2)^2 - 4(u^2y^2 + y^2z^2 + z^2u^2) = 0.$$

This equation will be identical with (1) provided

$$a = -2(u^2 + y^2 + z^2), \quad (2)$$

$$b = -8\,uyz, \text{ or } uyz = -\frac{b}{8}, \quad (3)$$

and $$c = (u^2 + y^2 + z^2)^2 - 4(u^2y^2 + y^2z^2 + z^2u^2). \quad (4)$$

By (2), $u^2 + y^2 + z^2 = -\frac{a}{2}$; and, by (3), $u^2y^2z^2 = \frac{b^2}{64}$.

Also, by (4), $$u^2y^2 + y^2z^2 + z^2u^2 = \frac{(u^2 + y^2 + z^2)^2 - c}{4}.$$

Then, $$u^2y^2 + y^2z^2 + z^2u^2 = \frac{\frac{a^2}{4} - c}{4} = \frac{a^2 - 4\,c}{16}.$$

By § 319, the cubic equation whose roots are u^2, y^2, and z^2 is

$$t^3 - (u^2 + y^2 + z^2)t^2 + (u^2y^2 + y^2z^2 + z^2u^2)t - u^2y^2z^2 = 0.$$

Putting for $u^2 + y^2 + z^2$, $u^2y^2 + y^2z^2 + z^2u^2$, and $u^2y^2z^2$, the values given above, this becomes

$$t^3 + \frac{a}{2}t^2 + \frac{a^2 - 4\,c}{16}t - \frac{b^2}{64} = 0. \quad (5)$$

If l, m, and n represent the roots of this equation, we have $u^2 = l$, $y^2 = m$, and $z^2 = n$; or, $u = \pm\sqrt{l}$, $y = \pm\sqrt{m}$, $z = \pm\sqrt{n}$.

Now $x = u + y + z$; and since each of the numbers u, y, and z has two values, apparently x has *eight* values.

But by (3), the product of the three terms whose sum is a value of x must be $-\frac{b}{8}$.

Hence, the only values of x are, when b is *positive*,

$$-\sqrt{l}-\sqrt{m}-\sqrt{n},\quad -\sqrt{l}+\sqrt{m}+\sqrt{n},$$

$$\sqrt{l}-\sqrt{m}+\sqrt{n}, \text{ and } \sqrt{l}+\sqrt{m}-\sqrt{n};$$

and when b is *negative*,

$$\sqrt{l}+\sqrt{m}+\sqrt{n},\quad \sqrt{l}-\sqrt{m}-\sqrt{n},$$

$$-\sqrt{l}+\sqrt{m}-\sqrt{n}, \text{ and } -\sqrt{l}-\sqrt{m}+\sqrt{n}.$$

Equation (5) is called the *auxiliary cubic* of (1).

Ex. Solve the equation

$$x^4 - 46x^2 - 24x + 21 = 0.$$

Here, $a = -46,\ b = -24,\ c = 21.$

Whence, $\frac{a^2 - 4c}{16} = 127$, and $\frac{b^2}{64} = 9.$

Then the auxiliary cubic is

$$t^3 - 23t^2 + 127t - 9 = 0.$$

By the method of § 356, one value of t is 9.

Dividing the first member by $t - 9$, the depressed equation is

$$t^2 - 14t + 1 = 0.$$

Solving, $t = 7 \pm \sqrt{49 - 1} = 7 \pm 4\sqrt{3}.$

Proceeding as in § 193, we have

$$\sqrt{(7 \pm 4\sqrt{3})} = \sqrt{(4 \pm 2\sqrt{12} + 3)} = 2 \pm \sqrt{3}.$$

Then since b is negative, the four values of x are

$$3 + 2 + \sqrt{3} + 2 - \sqrt{3},\quad 3 - 2 - \sqrt{3} - 2 + \sqrt{3},$$

$$-3 + 2 + \sqrt{3} - 2 + \sqrt{3}, \text{ and } -3 - 2 - \sqrt{3} + 2 - \sqrt{3}.$$

That is, $7,\ -1,\ -3 + 2\sqrt{3}$, and $-3 - 2\sqrt{3}$.

EXERCISE 93

Solve the following:

1. $x^4 - 60x^2 + 80x + 384 = 0.$

2. $x^4 - 44x^2 + 16x + 192 = 0.$

3. $x^4 - 40x^2 + 64x + 128 = 0.$

4. $x^4 - 54x^2 - 216x - 243 = 0.$

INCOMMENSURABLE ROOTS

373. We will now show how to find the approximate numerical values of those roots of an equation which are not commensurable (§ 355).

374. Horner's Method of Approximation.

Let it be required to find the approximate value of the root between 3 and 4 of the equation

$$x^3 - 3x^2 - 2x + 5 = 0.$$

We first transform the equation into another whose roots shall be respectively those of the first diminished by 3, by the second method explained in § 332.

The operation is conveniently performed by Synthetic Division (§ 108).

$$\begin{array}{lccccl}
 & 1 & -3 & -2 & +5 & \underline{|3} \\
 & & 3 & 0 & -6 & \\
\text{1st quotient,} & 1 & 0 & -2, & -1 & \text{1st Rem.} \\
 & & 3 & 9 & & \\
\text{2d quotient,} & 1 & 3, & 7 & \text{2d Rem.} & \\
 & & 3 & & & \\
 & & 6, & \text{3d Rem.} & &
\end{array}$$

The transformed equation is $y^3 + 6y^2 + 7y - 1 = 0.$ (1)

We know that equation (1) has a root between 0 and 1.

If, then, we neglect the terms involving y^3 and y^2, we may obtain an approximate value of y by solving the equation $7y - 1 = 0$; thus, approximately, $y = .1$ and $x = 3.1$.

Transforming (1) into an equation whose roots shall be respectively those of (1) diminished by .1, we have

$$\begin{array}{rrrrl}
1 & +6 & +7 & -1 & \underline{|.1} \\
 & \underline{.1} & \underline{.61} & \underline{.761} & \\
 & 6.1 & 7.61 & -\ .239 & \\
 & \underline{.1} & \underline{.62} & & \\
 & 6.2 & 8.23 & & \\
 & \underline{.1} & & & \\
 & 6.3 & & &
\end{array}$$

The transformed equation is

$$z^3 + 6.3\,z^2 + 8.23\,z - .239 = 0. \qquad (2)$$

Neglecting the z^3 and z^2 terms, we have, approximately,

$$z = \frac{.239}{8.23} = .02.$$

Thus, the value of x to two places of decimals is 3.12.

The work is usually arranged in the following form, the coefficients of the successive transformed equations being denoted by (1), (2), (3), etc.

$$\begin{array}{rrrrrrl}
1 & -3 & & -2 & & +5 & \underline{|3.128} \\
 & \underline{3} & & \underline{0} & & \underline{-6} & \\
 & 0 & & -2 & (1) & -1 & \\
 & \underline{3} & & \underline{9} & & \underline{.761} & \\
 & 3 & (1) & 7 & (2) & -\ .239 & \\
 & \underline{3} & & \underline{.61} & & \underline{.167128} & \\
(1) & 6 & & 7.61 & (3) & -\ .071872 & \\
 & \underline{.1} & & \underline{.62} & & & \\
 & 6.1 & (2) & 8.23 & & & \\
 & \underline{.1} & & \underline{.1264} & & & \\
 & 6.2 & & 8.3564 & & & \\
 & \underline{.1} & & \underline{.1268} & & & \\
(2) & 6.3 & (3) & 8.4832 & & & \\
 & \underline{.02} & & & & & \\
 & 6.32 & & & & & \\
 & \underline{.02} & & & & & \\
 & 6.34 & & & & & \\
 & \underline{.02} & & & & & \\
(3) & 6.36 & & & & &
\end{array}$$

Dividing .071872 by 8.4832, we have .008 +, and the value of x to three places of decimals is 3.128.

The process can be continued until the root has been found to any desired degree of precision.

We derive from the above the following rule for finding the approximate value of a positive incommensurable root:

Find by § 346, or by Sturm's Theorem (§ 350), the integral part of the root. (Compare § 347.)

Transform the given equation into another whose roots shall be respectively those of the first diminished by this integral part.

Divide the absolute value of the last term of the transformed equation by the absolute value of the coefficient of the first power of the unknown number, and write the approximate value of the result as the next figure of the root.

Transform the last equation into another whose roots shall be respectively those of the first diminished by the figure of the root last obtained, and divide as before for the next figure of the root; and so on.

In practice, the work may be contracted by dropping such decimal figures from the right of each column as are not needed for the required degree of accuracy.

In determining the integral part of the root, it will be found convenient to construct the graph of the first member of the given equation.

375. To find an approximate value of a *negative* incommensurable root, change the signs of the alternate terms of the equation commencing with the second (§ 326), and find the corresponding positive incommensurable root of the transformed equation.

The result with its sign changed will be the required negative root.

376. In finding any particular root-figure by the method of § 374, we are liable, especially in the first part of the process, to get too great a result; the same thing occasionally happens when extracting square or cube roots of numbers.

Such an error may be discovered by observing the signs of the last two terms of the next transformed equation; for since each root-figure obtained as in § 374 must be *positive*, the last two terms of the transformed equation must be of *opposite sign.*

If this is not the case, the last root-figure must be diminished until a result is obtained which satisfies this condition.

Let it be required, for example, to find the root between 0 and -1 of the equation $x^3 + 4x^2 - 9x - 5 = 0$.

Changing the signs of the alternate terms commencing with the second (§ 326), we have to find the root between 0 and 1 of the equation

$$x^3 - 4x^2 - 9x + 5 = 0.$$

Dividing 5 by 9, we have .5 suggested as the first root-figure; but it will be found that in this case the last two terms of the first transformed equation are -12.25 and $-.375$.

This shows that .5 is too great; we then try .4, and find that the last two terms of the first transformed equation are of opposite sign.

The work of finding the first three root-figures is shown below.

1	− 4		− 9		+ 5	.469
	.4		− 1.44		− 4.176	
	− 3.6		− 10.44	(1)	.824	
	.4		− 1.28		− .713064	
	− 3.2	(1)	− 11.72	(2)	.110936	
	.4		− .1644			
(1)	− 2.8		− 11.8844			
	.06		− .1608			
	− 2.74	(2)	− 12.0452			
	.06					
	− 2.68					
	.06					
(2)	− 2.62					

The required root is $-.469$, to three places of decimals.

377. Sometimes too *small* a number is suggested for the first root-figure.

Let it be required, for example, to find the root between 0 and 1 of the equation

$$x^3 - 2x^2 + 3x - 1 = 0.$$

Dividing 1 by 3, we have .3 suggested as the first root-figure.

1	-2	$+3$	-1	$\lfloor .3$
	.3	$-.51$	.747	
	-1.7	2.49	$-.253$	
	.3	$-.42$		
	-1.4	2.07		
	.3			
	-1.1			

The number suggested by the next division is greater than .1; showing that too small a root-figure has been taken.

378. If the coefficient of the first power of the unknown number in any transformed equation is zero, the next figure of the root may be obtained by *dividing the absolute value of the last term by the absolute value of the coefficient of the square of the unknown number, and then taking the square root of the result.*

For if the transformed equation is $y^3 + ay^2 + b = 0$, it is evident that, approximately, $ay^2 + b = 0$, or $y = \sqrt{-\frac{b}{a}}$.

We proceed in a similar manner if any number of consecutive terms immediately preceding the last term are zero.

Horner's method may be used to find any root of a number approximately; for to find the nth root of a is the same thing as to solve the equation $x^n - a = 0$.

379. If an equation has two or more roots which have the same integral part, the first decimal root-figure of each must be obtained by the method of § 346, or by Sturm's Theorem.

If two or more roots have the same integral part, and also the same first decimal root-figure, the second decimal root-figure of each must be obtained by the method of § 346, or by Sturm's Theorem; and so on.

Horner's method may be used to determine successive figures in the *integral*, as well as in the decimal, portion of the root.

If all but one of the roots of an equation are known, the remaining root may be found by changing the sign of the coefficient of the second term of the given equation, and subtracting the sum of the known roots from the result (§ 321).

EXERCISE 94

Find the root between:

1. 1 and 2, of $x^3 - 9x^2 + 23x - 16 = 0$.
2. 4 and 5, of $x^3 - 4x^2 - 4x + 12 = 0$.
3. 0 and -1, of $x^3 + 8x^2 - 9x - 12 = 0$.
4. -2 and -3, of $x^3 - 3x^2 - 9x + 4 = 0$.
5. 3 and 4, of $x^3 - 6x^2 + 15x - 19 = 0$.
6. 0 and 1, of $x^4 + x^3 + 2x^2 - x - 1 = 0$.
7. 2 and 3, of $x^4 - 3x^3 + 4x - 5 = 0$.
8. -1 and -2, of $x^4 - 2x^3 - 3x^2 + x - 2 = 0$.

Find all the real roots of the following:

9. $x^3 + 2x^2 - x - 1 = 0$.
10. $x^3 - 2x^2 - 7x - 1 = 0$.
11. $x^3 - 5x^2 + 2x + 6 = 0$.
12. $x^4 + 2x^3 - 5 = 0$.
13. $x^3 - x^2 + 2x - 1 = 0$.
14. $x^4 - 6x^2 + 11x + 21 = 0$.

Find the approximate values of the following:

15. $\sqrt[3]{3}$.
16. $\sqrt[3]{21}$.
17. $\sqrt[4]{7}$.

380. We may now give general directions for finding the real roots of any equation of the form

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

with integral numerical coefficients:

1. Determine by Descartes' Rule (§ 335) limits to the number of positive and negative roots.

2. Find all the commensurable roots, if any, as explained in § 356.

3. If possible, locate the incommensurable roots by the method of § 346.

4. If the incommensurable roots are not all located in this way, apply Sturm's Theorem (§ 350), observing that, if the first member and its first derivative have a common factor, the given equation has multiple roots (§ 343).

5. Approximate to the decimal portions of the incommensurable roots by Horner's method (§ 374).

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